Singapore International Mathematical Olympiad 2003
National Team Training

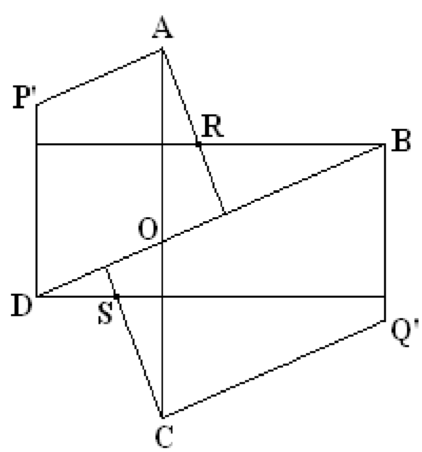
Geometry

(8) Let P and Q be the feet of the perpendiculars from the orthocenter on triangle ABC onto the internal and external bisectors of $\angle A$. Assume that $\angle A$ is not a right angle, show that the line through P and Q bisects BC.

Solution. Let H be the orthocenter and consider the circle $\mathcal{C}$ with diameter AH. Observe that P, S, Q, D all lie on this circle. Since $\angle PAQ$ is a right angle, PQ is another diameter of $\mathcal{C}$. Also $\angle DAQ = \angle SAQ$, so the arcs DQ and SQ on $\mathcal{C}$ have the same length. Thus the diameter PQ must be the perpendicular bisector of the segment DS. Now consider the circle $\mathcal{C}'$ with diameter BC. S and D both lie on $\mathcal{C}'$. Hence PQ, which is the perpendicular bisector of the chord DS of $\mathcal{C}'$, must pass through the center of $\mathcal{C}'$, which is the midpoint of BC.

(9) Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD intersect at O. If P and Q are the centroids of triangles AOD and BOC respectively, and R and S are the orthocenters of triangles AOB and DOC respectively, show that $PQ \perp RS$.

Solution. Construct P' and Q' so that $P'AOD$ nd $Q'COB$ are parallelograms. We first claim that PQ is parallel to $P'Q'$. In fact, using vectors, $OP' = OA + OD$ and $OP = (OA + OD)/3$. So $OP' = 3OP$. Similarly, $OQ' = 3OQ$. Thus $P'Q' = OQ' - OP' = 3(OQ - OP) = 3PQ$. This proves the claim.



Hence it suffices to show that RS is perpendicular to $P'Q'$. Draw all four lines through B and D parallel and perpendicular to AC. From here, I use coordinate geometry, you may be able to find a better way. Set up a coordinate system with D as the origin and DP' as the y-axis. Set $A = (a, b)$, $B = (c, d)$ and $C = (a, e)$. We find that

$$P' = (0, b - \frac{ad}{c}) \quad R = (a + \frac{d(b-d)}{c}, d)$$
$$S = (a + \frac{ed}{c}, 0) \quad Q' = (c, e + \frac{(c-a)d}{c}).$$

Hence the slope of $P'Q'$ is $(e + d - b)/c$ and the slope of RS is $c/(b - d - e)$. This shows that $P'Q'$ and RS are perpendicular.

(10) Let E be an interior point on the median AD of acute triangle ABC. Label the foot of the perpendicular from E onto BC as F. From and interior point M of the segment EF, drop perpendiculars onto the sides AB and AC and let the feet of these perpendiculars be P and N respectively. Show that the angle bisectors of $\angle PMN$ and $\angle PEN$ are either parallel or coincide.

Solution. Observe that $APMN$ are concyclic. Hence $\angle PMN = 180° - \angle A$. Therefore, the bisector of $\angle PMN$ makes an angle of $90° - \angle A/2$ with the segment PM, which impies that it makes an angle of $\angle A/2$ with the line AP. Thus the bisector of $\angle PMN$ is parallel to the bisector of $\angle A$. We now show that the bisector of $\angle PEN$ is also parallel to the bisector of $\angle A$. Let the line through E parallel to BC meet the sides AB and AC at H and K respectively. Since E lies on the median from A, $HE = EK$. Also, $EHPM$ are concyclic and $EKNM$ are concyclic. Therefore,

$$\angle EPH = \angle EMH = \angle EMK = \angle ENK.$$

Considering the quadrilateral $APEN$,

$$360° - \angle PEN + \angle A + \angle EPH + \angle ENK = 360°.$$

It follows that $\angle PEN = \angle A + 2\angle EPH$. Thus the bisector of $\angle PEN$ makes and angle of $\angle EPH + \angle A/2$ with PE. Hence it is parallel to the bisector of $\angle A$, as required.