Training problems 1 April 2003

8. In a group of interpreters each one speaks one or several languages, 24 of them speak Japanese, 24 Chinese and 24 English. Prove that it is possible to select a subgroup in which exactly 12 interpreters speak Japanese, exactly 12 speak Chinese and exactly 12 speak English.

Solution: Suppose that in a group of interpreters n speak Japanese, n speak Chinese and n speak English. Denote these groups by A, B, C. Put $p = |A \cap B^c \cap C^c|$, $q = |A^c \cap B \cap C^c|$, $r = |A^c \cap B^c \cap C|$, $a = |A^c \cap B \cap C|$, $b = |A \cap B^c \cap C|$, $c = |A \cap B \cap C^c|$, $d = |A \cap B \cap C|$.

A group of interpreters is called a k-group if exactly k interpreters speak Japanese, exactly k speak Chinese and exactly k speak English.

We shall prove by induction on n, that for $n \geq 2$, it's possible to find a 2-group inside an n-group.

When $n = 2$, it's trivially true. Now suppose $n > 2$ is an integer and that for each k, $2 \leq k < n$, the result is true.

1. $a, b, c > 0$: It's enough to select one interpreter from each of the sets:

$$A^c \cap B \cap C, \quad A \cap B^c \cap C, \quad A \cap B \cap C^c.$$

2. $p, q, r > 0$: Select one from each of the sets:

$$A^c \cap B^c \cap C, \quad A^c \cap B \cap C^c, \quad A \cap B^c \cap C^c$$

and then apply induction on the remaining people.

3. $d > 0$: It's enough to select one from $A \cap B \cap C$ apply the induction hypothesis to the remaining group.

4. None of the above hold: One of a, b, c is 0, say $a = 0$; $d = 0$ and one of p, q, r is 0. We have $q + c = b + r = p + b + c = n$. If $q = 0$, then $c = r = n$ and $p = b = 0$. Thus $c > 0, r > 0$. We can choose one from each of $A \cap B \cap C^c$, $\quad A^c \cap B^c \cap C$ and then apply the induction hypothesis. The case $r = 0$ is similar. The final case if $p = 0$ and $r, q > 0$. Then since $b + c = n$, one of them is positive, say $b > 0$. Then choose one from each of $A \cap B^c \cap C$, $\quad A^c \cap B \cap C^c$ and then apply the induction hypothesis.

Thus from $n = 24$, we can choose k as long as k is an even number less than 24.

Solution: *2nd soln by Colin.* Let L_1, L_2, L_3 be the three languages. Divide the interpreter into 7 groups $S(l_1, l_2, l_3)$ where $l_i = 1$ if the people from the group speak L_i and $l_i = 0$ otherwise. Let $a_1, \ldots, a_7$ be,respectively, the number of people in the $S(1,0,0)$, $S(0,1,0)$, $S(0,0,1)$, $S(0,1,1)$, $S(1,0,1)$, $S(1,1,0)$, $S(1,1,1)$.

We shall prove that for any $n \geq 2$, it's possible to find a 2-group inside an n-group.

We have 3 equations by considering the interpreters who can speak each of the languages in turn.

$$a_1 + a_5 + a_6 + a_7 = n$$

$$a_2 + a_4 + a_6 + a_7 = n$$

$$a_3 + a_4 + a_6 + a_7 = n$$

Without loss of generality, we can assume that $a_1 \le a_2 \le a_3$. The solutions are of the form

$$(a_1, \ldots, a_7) = (a, a+b, a+c, d, d+b, d+c, n-(a+b+c+2d))$$

for (independent) nonnegative integers a, b, c, d.

The set $\{S(1,0,0), S(0,1,0), S(0,0,1)\}$ gives a 1-groups. The set $\{S(0,1,0), S(1,0,1)\}$ gives b 1-groups. The set $\{S(0,0,1), S(1,1,0)\}$ gives c 1-groups. The set $S(1,1,1)$ gives $24-(a+b+c+2d)$ 1-groups. The set $\{S(0,1,1), S(1,0,1), S(1,1,0)\}$ gives d 2-groups.

If $(a)+(b)+(c)+(24-(a+b+c+2d)) = n-2d \ge 2$, then there are 2 1-groups which will combine to give a 2-group. Otherwise, $n-2d \le 1$, or $2d \ge n-1$ or $d \ge 1$. We still have a 2-group.

Apply this result to the case $n = 24$, we have a 2-group. Remove this 2-group, we are left with the case with $n = 22$. Continuing this way, we can find 6 distinct 2-groups and they combine to give a 12-group as desired.

9. Points $P_1, \ldots, P_n$ are placed inside or on the boundary of a disk of radius 1 in such a way that the minimum distance d_n between any two of these points has its largest possible value D_n. Calculate D_n for $n = 2, \ldots, 7$. Justify your answers.

Solution: Suppose $n \le 6$. Decompose the disk by its radii into n congruent regions so that one of the points is on one of the radii. Then there is one region (including its boundary) which contains 2 of the points. Since the distance between any two points in a region is at most $2\sin\pi/n$, then $d_n \le 2\sin\pi/n$. If points P_j are placed in the vertices of regular n-gon inscribed in the boundary of the disk, then $d_n = 2\sin\pi/n$. Therefore $D_n = 2\sin\pi/n$.

For $n = 7$, we have $D_7 \le D_6 = 1$. If one of the given points is placed in the center of the disk and if the other 6 points are placed at the vertices of the regular hexagon inscribed in the boundary of the disk, then $d_7 = 1$. Thus $D_7 = 1$.

10. Prove that in any triangle, a line passing through the incentre cuts the perimeter of the triangle in half if and only if it halves the area of the triangle.

Solution: Let ABC be the triangle and O, r denote t he incentre and inradius. Let l be a line passing through O. It intersects one side, say BC, at an interior point. Without loss of generality, let it intersect the side AC at P. (Note that P may coincide with A.) Let $x = PC$, $y = QC$. For a triangle XYZ, denote its area by $[XYZ]$. Then

$$[ABC] = \frac{r(a+b+c)}{2}, \qquad [CPQ] = \frac{r(x+y)}{2}$$

the latter because the altitudes of $\triangle OCP, \triangle OCQ$ from O are both r. The line l halves the area iff

$$a+b+c = 2(x+y)$$

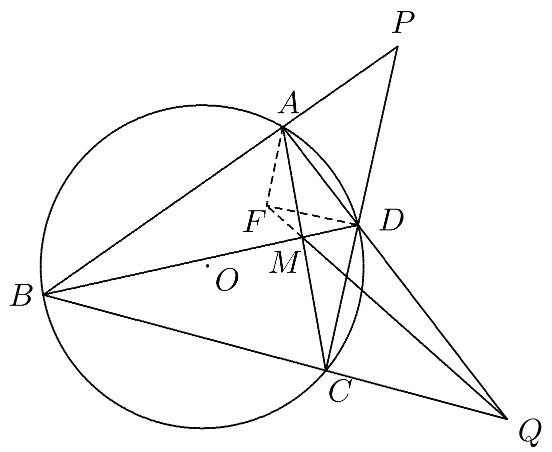
iff l halves the perimeter.

11. Nine positive integers $a_1 < a_2 < \cdots < a_9$ are such that all the sums (of at least one and at most nine different terms) that can be made up of them are different. Prove that $a_9 > 100$.

Solution: Assume that $a_9 \le 100$. Let S be the set of those sums $\ge a_4$ of at most 5 terms out of $a_1, \ldots, a_8$. The number of sums of at most 5 terms is $\binom{8}{1} + \cdots + \binom{8}{5} = 218$. Those that can be less than a_4 are made up of a_1, a_2, a_3 and there are at most 7 of them. Thus $|S| \geq 211$. The greatest sum in S is $a_4 + \cdots + a_8 < a_4 + 4a_9$ and therefore all the sums are in $[a_4, a_4 + 4a_9]$. The inequality $|S| \geq 2a_9$ implies that there are 3 numbers which are congruent mod a_9. Thus 2 of them must have a difference of a_9, a contradiction.

12. The quadrilateral $ABCD$ inscribes in a circle with centre O. Let BA meet CD at P, AD meet BC at Q and AC meet BD at M. Show that O is the orthocentre of triangle PQM.

Solution: Let R be the radius of the circle. As $\angle QMD > \angle CBD = \angle DAM$, one can extend QM to QF such that $\angle FAD = \angle QMD$. Then A, D, M, F are concyclic. Also $\angle QBD = \angle DAM = \angle DFM$ so that B, F, D, Q are concyclic.



Thus, $QM \cdot QF = QD \cdot QA = QO^2 - R^2$, and $QM \cdot MF = MB \cdot MD = R^2 - MO^2$. Subtracting, we get $QM(QF - MF) = QO^2 + MO^2 - 2R^2$. That is $QM^2 = QO^2 + MO^2 - 2R^2$ Similarly, $PM^2 = PO^2 + MO^2 - 2R^2$. Subtracting again, we have $PM^2 - QM^2 = PO^2 - QO^2$.

It follows from this that OM is perpendicular to PQ. To see this, suppose the extension of OM meets PQ at E and $\angle PEM > \angle QEM$. By cosine rule, $PM^2 = EP^2 + ME^2 - 2EP \cdot ME \cos \angle PEM > EP^2 + ME^2$. Similarly, $PO^2 > EP^2 + OE^2, QM^2 < EQ^2 + ME^2$ and $QO^2 < EQ^2 + OE^2$. Subtracting, we obtain $PM^2 - QM^2 > EP^2 - EQ^2 > PO^2 - QO^2$, a contradiction.

Similarly, PM is perpendicular to OQ and QM is perpendicular to OP. Therefore, O is the orthocentre of triangle PQM.

(*Alternate Solution*) Let OM meet PQ at W. From W draw tangents to the circle touching it at H and G Then H, M, G are collinear as M is the pole of PQ. Let HG meet PQ at Z. Then the cross ratio $(H, G; M, Z) = -1$. Since WM bisects $\angle HWG$, we have OW or MW is perpendicular to PQ.

13. Suppose $a_1, a_2, \cdots, a_n$ are $n \geq 3$ positive numbers such that $(a_1^2 + a_2^2 + \cdots + a_n^2)^2 > (n-1)(a_1^4 + a_2^4 + \cdots + a_n^4)$. Prove that any three such $a_i's$ form the lengths of the sides of a triangle.

Solution: First we prove the assertion when $n = 3$. Let's write the numbers as a, b, c. They satisfy the inequality : $(a^2 + b^2 + c^2)^2 > 2(a^4 + b^4 + c^4)$. We may assume without loss of generality that $a \geq b \geq c$. Then

$$0 \leq (a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4) = -[a^2 - (b + c)^2][a^2 - (b - c)^2].$$

Thus, $|b - c| < a < |b + c|$. Therefore, a, b, c are the lengths of the sides of a triangle.

In the general case, we can simply show that a_1, a_2, a_3 are the lengths of the sides of a triangle. Using Cauchy-Schwarz inequality, we have $(n-1)(a_1^4 + a_2^4 + \cdots + a_n^4) \leq \left(1 \cdot \frac{a_1^2+a_2^2+a_3^2}{2} + 1 \cdot \frac{a_1^2+a_2^2+a_3^2}{2} + 1 \cdot a_4^2 + \cdots + 1 \cdot a_n^2\right)^2 \leq (n-1)\left[2\left(\frac{a_1^2+a_2^2+a_3^2}{2}\right)^2 a_4^4 + \cdots + a_n^4\right]$. From this, we get $(a_1^2 + a_2^2 + a_3^2)^2 > 2(a_1^4 + a_2^4 + a_3^4)$. Using the case for $n = 3$, a_1, a_2, a_3 are the lengths of the sides of a triangle.