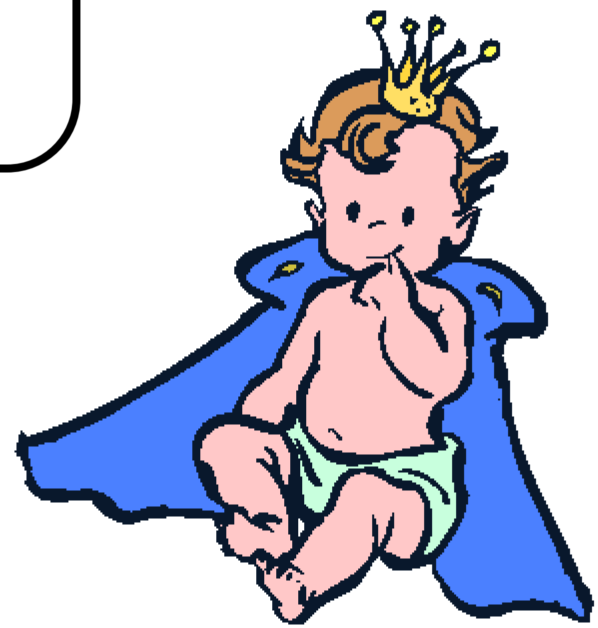
Domination in *Digraphs*



Koh Khee-Meng

matkohhkm@nus.edu.sg

Objectives

- Present some *fundamental* results on "***dominating sets***"
- Learn how to ***generalize*** or ***extend*** existing results
- Cultivate the ***Habit*** of ***Problem-Posing***
  — the first step towards doing ***research***

Digraph (Directed Graph)

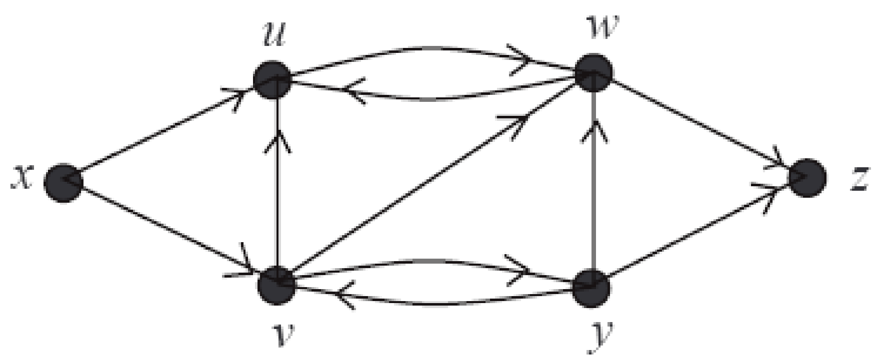
Digraph : $\boldsymbol{D} = (V, A)$

V = ***vertex set*** of $\boldsymbol{D}$

$= \{u, v, w, x, y, z\}$

A = ***arc set*** of $\boldsymbol{D}$

$= \{xu, xv, uw, wu, \cdots, yz\}$



- **Domination**

  u ***dominates*** x

- **Degrees**

  $od(v)$ = ***out*degree** of v = **3**

  $id(v)$ = ***in*degree** of v = **2**

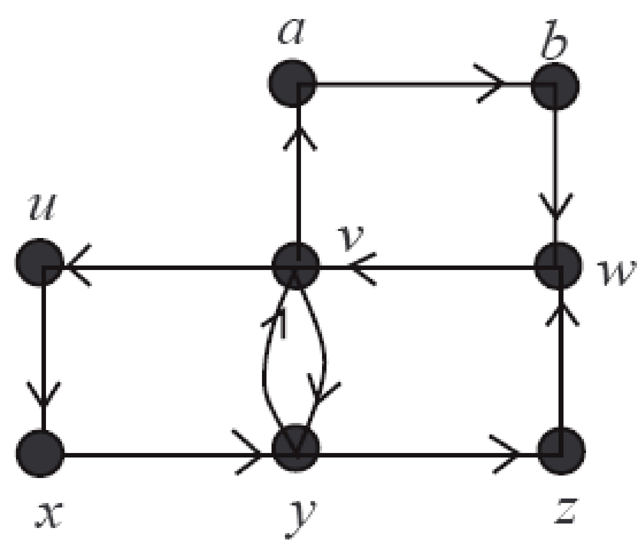
- **Distance**

  $d(a, z)$ = ***distance from*** a ***to*** z

  = the ***min***. no. of arcs traversed from a to z

  = **5**

Note $d(z, a)$ = **3**

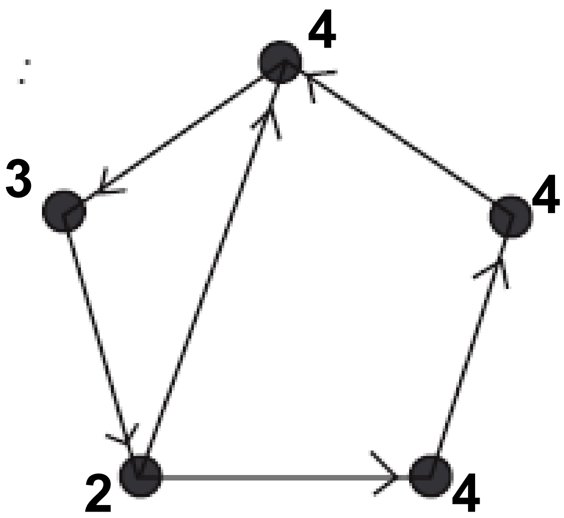


Dominating Vertices (*Kings*)

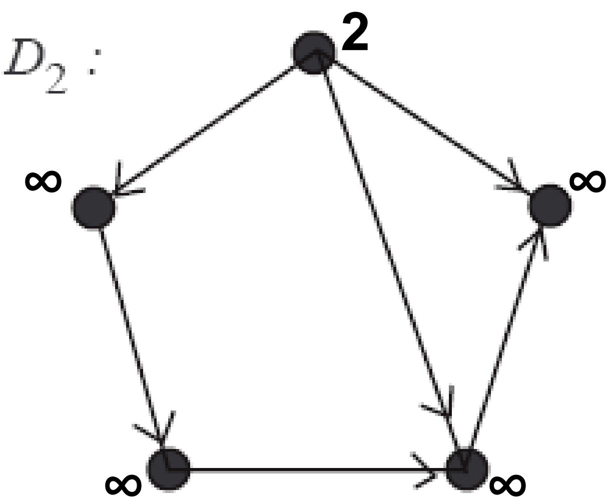
A vertex ***w*** in ***D*** is an ***r-king*** if

$d(w, v) \le r$ for all v in V.

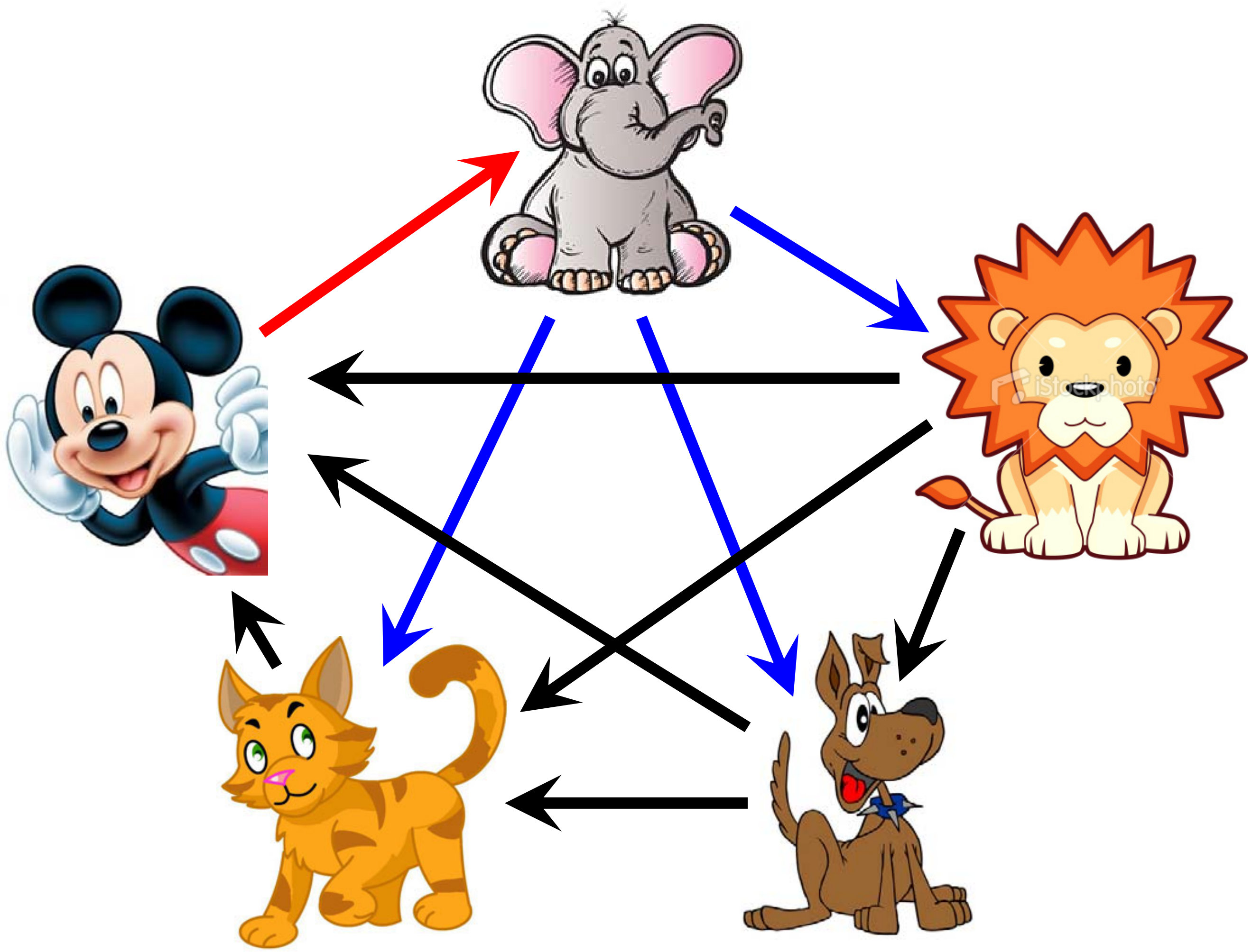
D_1 :



D_2 :

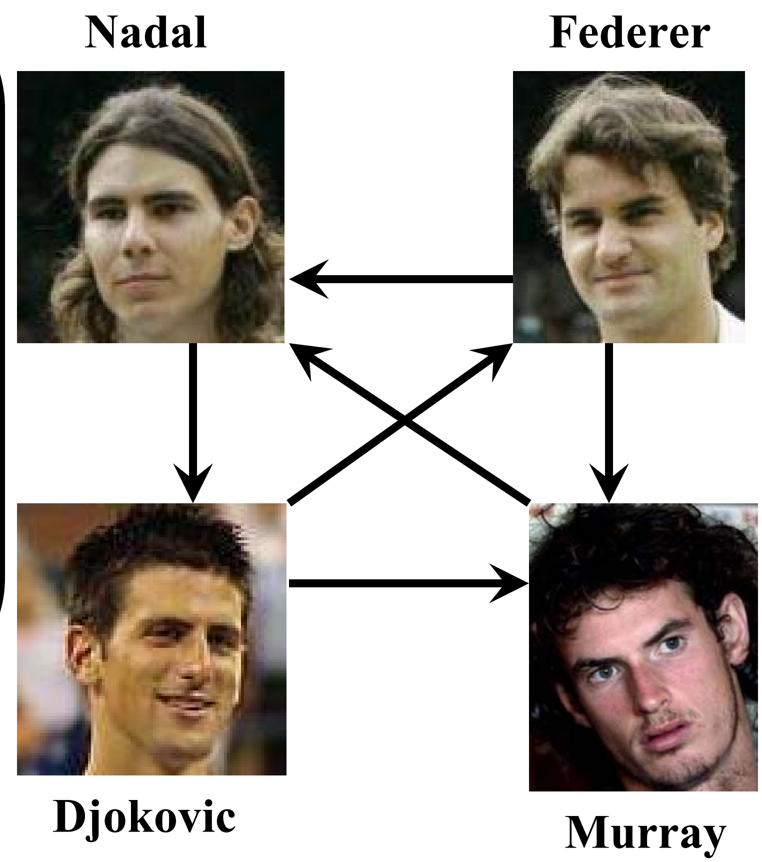


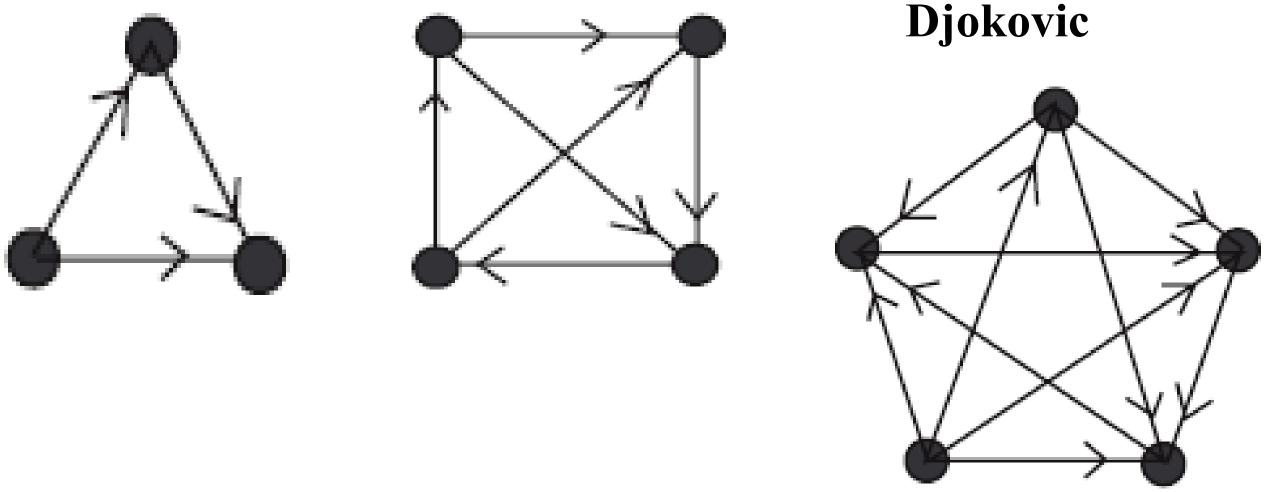
Landau (1909-1966)
- *Dominance Relations in* Animal Societies



Tournament

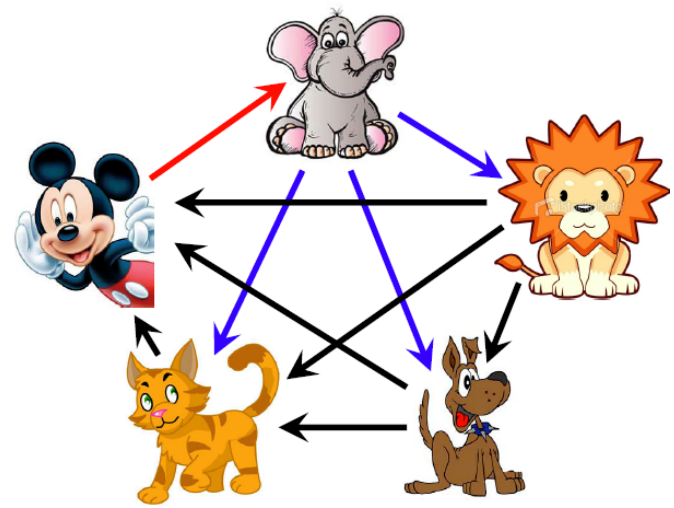
a digraph in which every two vertices are joined by ***exactly one*** arc.





***k*(*r*, *D*)**

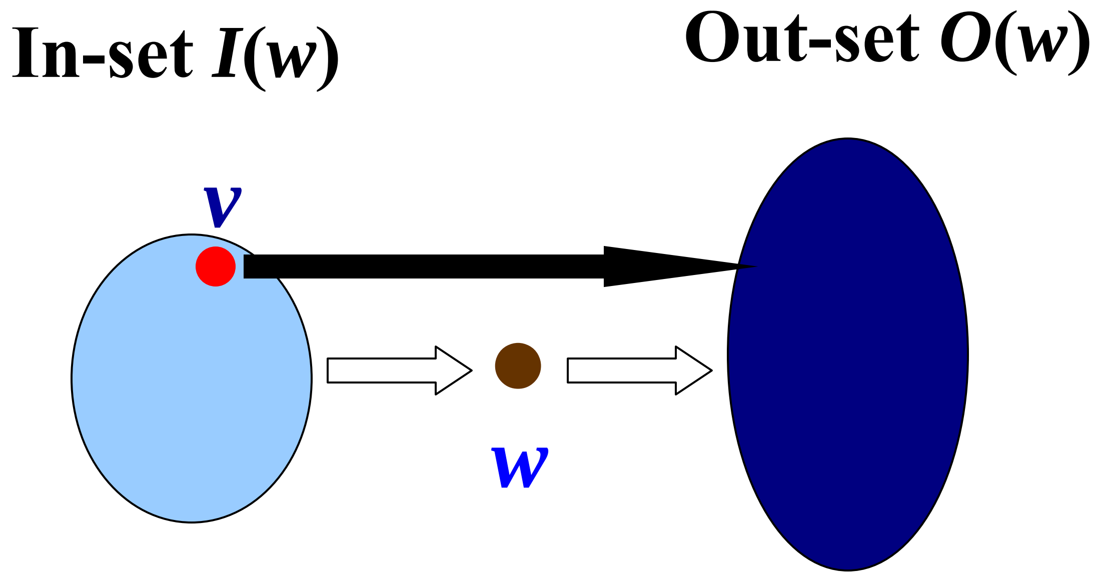
= # of ***r*-kings** in *D*



Landau (1953)

In any tournament T, any vertex with ***highest*** outdegree (**score**) is a **2-*king***.
Thus, $\boldsymbol{k(2, T) \geq 1}$.

Proof of Landau's **Observation**



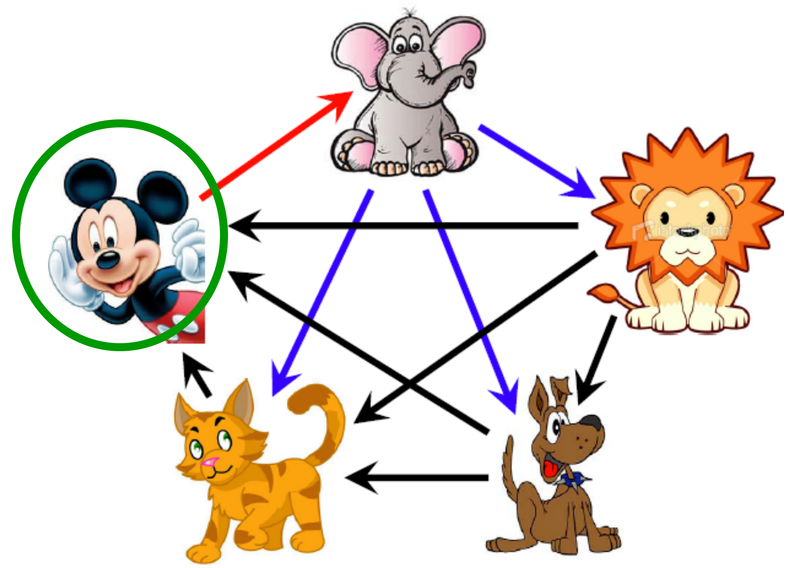
$$od(v) > od(w)$$

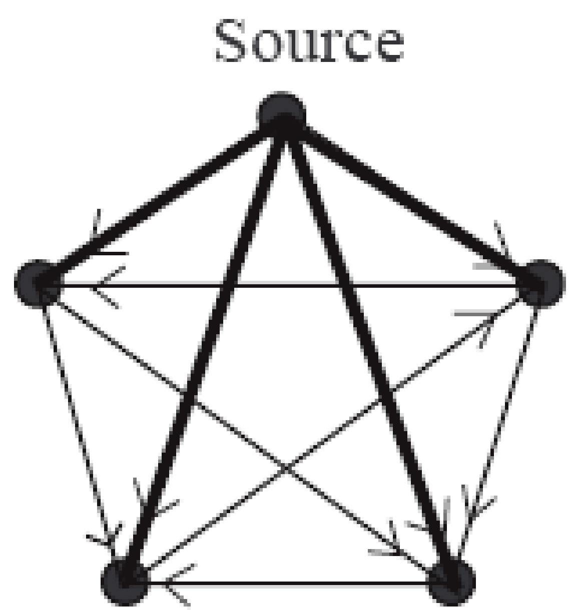
contradiction

Questions

- If ***w*** is a **2-*king***, must ***w*** have the ***highest score***?
- Are there tournaments ***T*** s.t. $\boldsymbol{k(2, T) = 1}$?

♣ A ***source*** is a vertex with zero indegree.





Every ***T*** with a ***source*** contains a ***unique*** 2-king.

- Is the ***converse*** of the above true?
- Under what conditions for ***T*** that
  $\boldsymbol{k(2, T) \geq 2}$?
- Are there tournaments ***T*** s.t.
  $\boldsymbol{k(2, T) = 2}$?
- What is the ***best lower bound*** for
  $\boldsymbol{k(2, T)}$ if ***T*** contains no source?

Moon's Observation (1962)

Let T be a tournament with no sources.

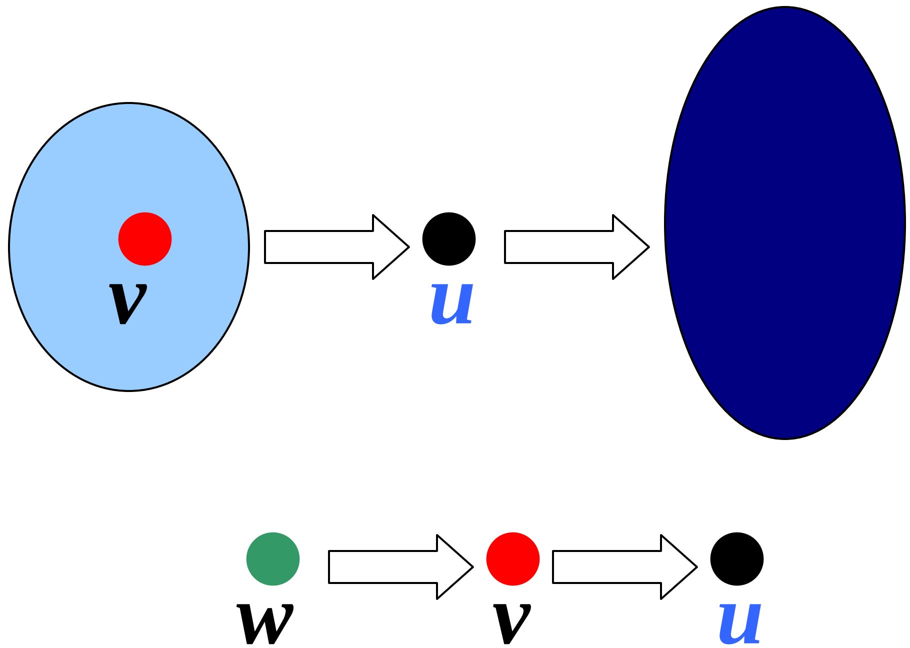
Then every 2-*king* is *dominated* by a 2-*king* in T.

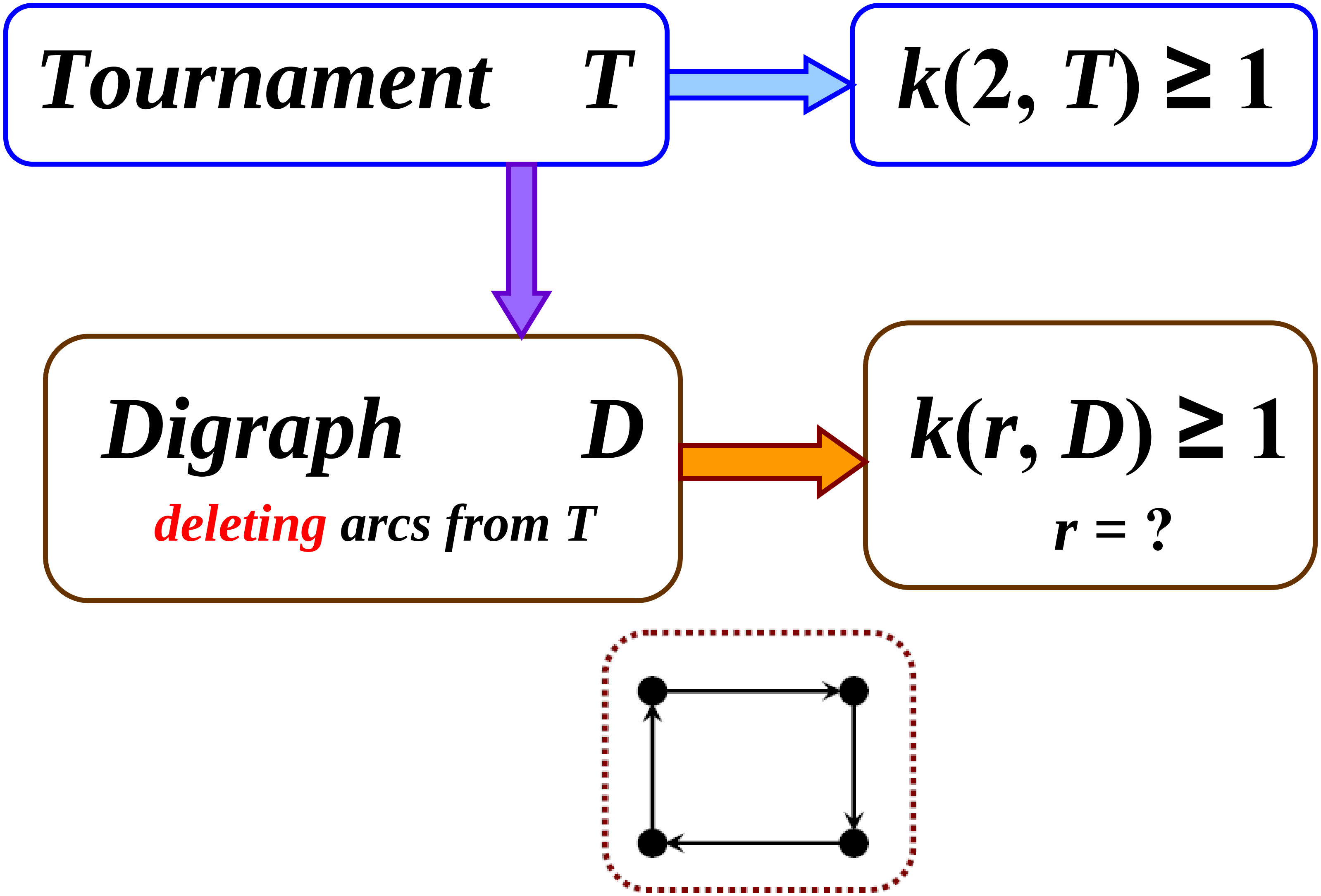
In particular,

$$k(2, T) \geq 3.$$

(***No*** tournament can have exactly **two 2-*kings***.)

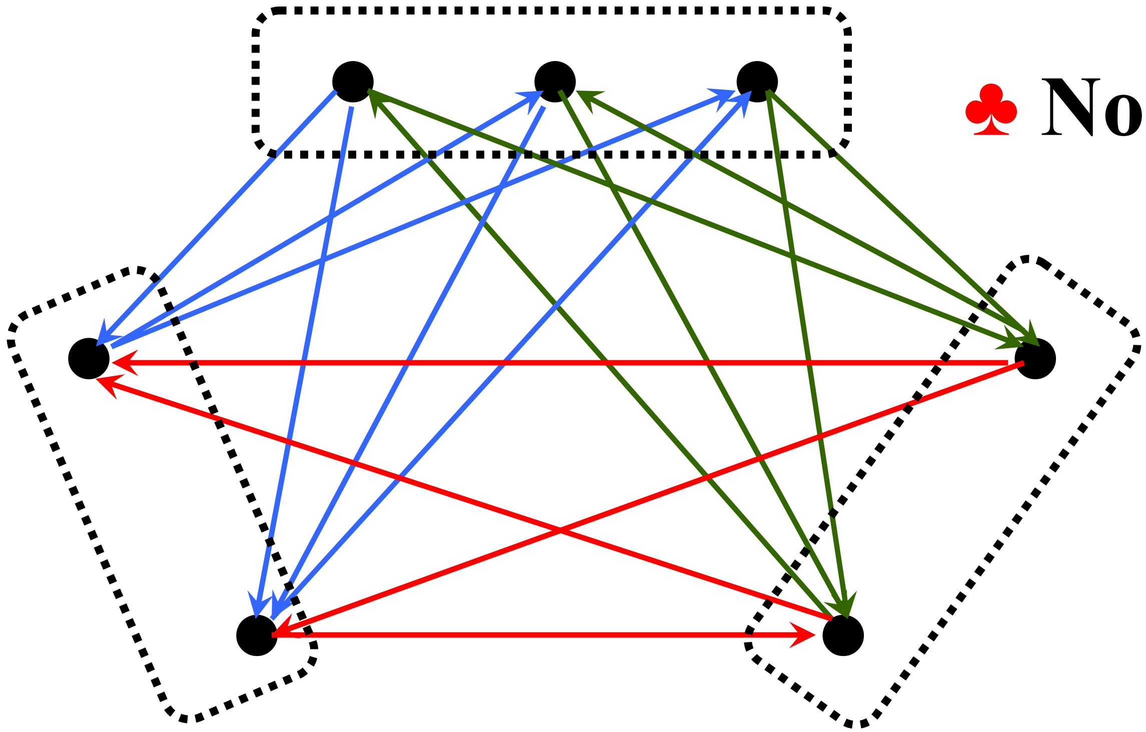
Proof of Moon's Observation




Tournament T
k(2, T) ≥ 1
Digraph D
deleting arcs from T
k(r, D) ≥ 1
r = ?

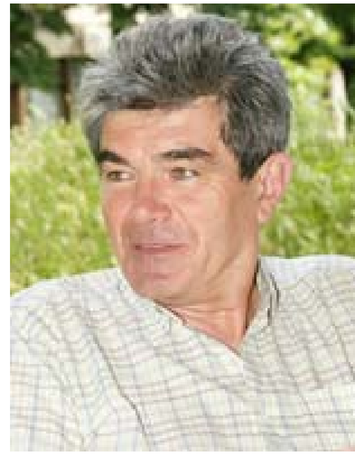
Multipartite Tournaments

3-partite tournament: ***T*(3,2,2)**



♣ **No sources**

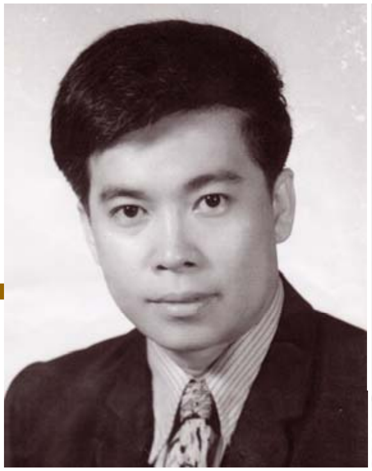
Petrovic & Thomassen (1991)



Let D be a *multipartite tournament* with *at most one source* . Then

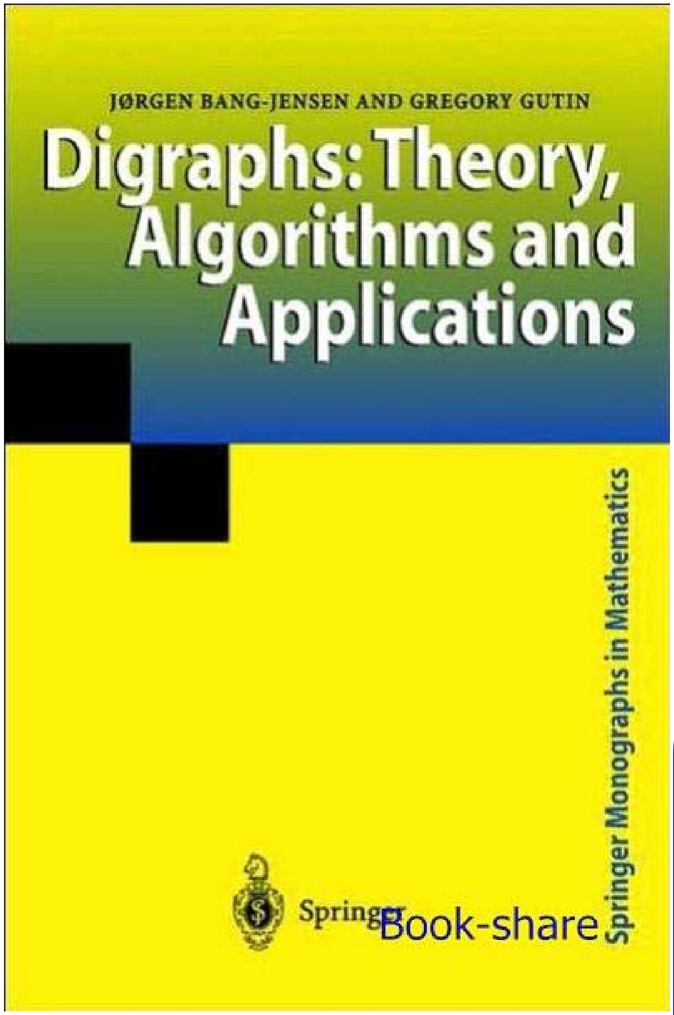
$$k(4, D) \geq 1.$$

Koh & Tan (1995)



Let D be an *n-partite tournament* with *no source*. Then

$$k(4, D) \geq \begin{cases} 4 & n = 2 \\ 3 & n \geq 3 \end{cases}$$



Thus, 4-kings are of particular interest in multipartite tournaments. In a number of papers, several authors investigate the minimum number of 4-kings in multipartite tournaments without sources. In our view, the above theorem is the most interesting result in this direction. [p.76]

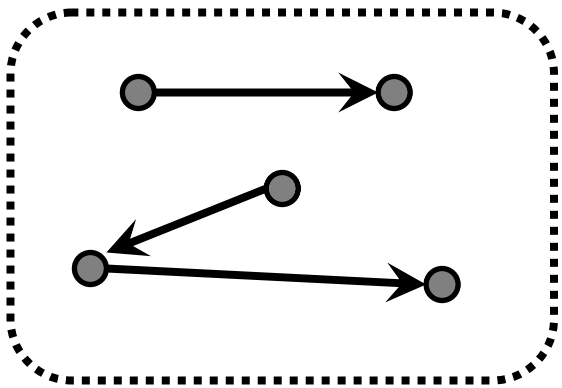
Individuals

Teams

Landau: T **tournament**

$$k(2, T) \geq 1.$$

Any 'team' version of Landau's result for general D **?**

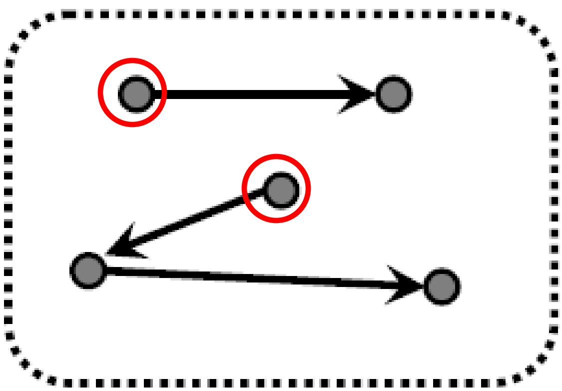


S $\subseteq$ *V* is an ***independent*** set if
no two vertices in ***S*** are joined by an arc in ***D*.**

K is a ***r-dominating set*** of ***D***

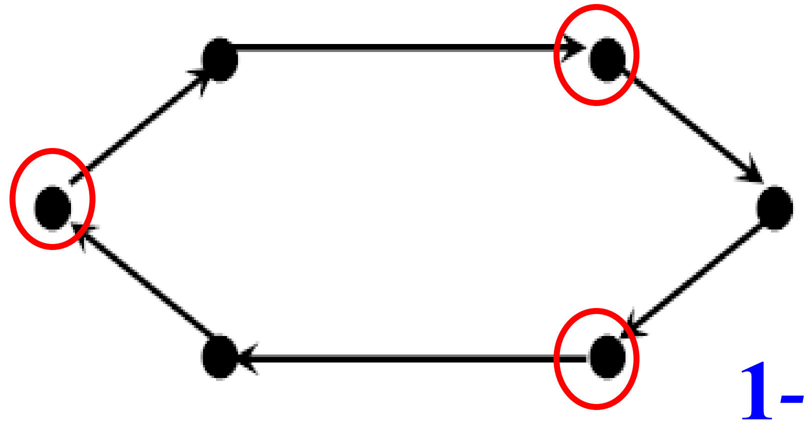
(i) ***K*** is ***independent*** &

(ii) every vertex in ***V*** \ ***K*** can be reached from ***a vertex*** in ***K*** *within* ***r steps***.

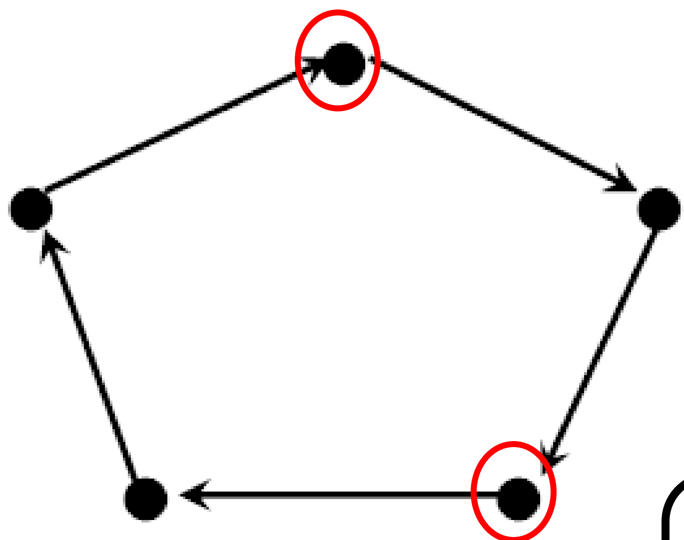


2-dominating set

***Dicycles*:**



1-dominating set



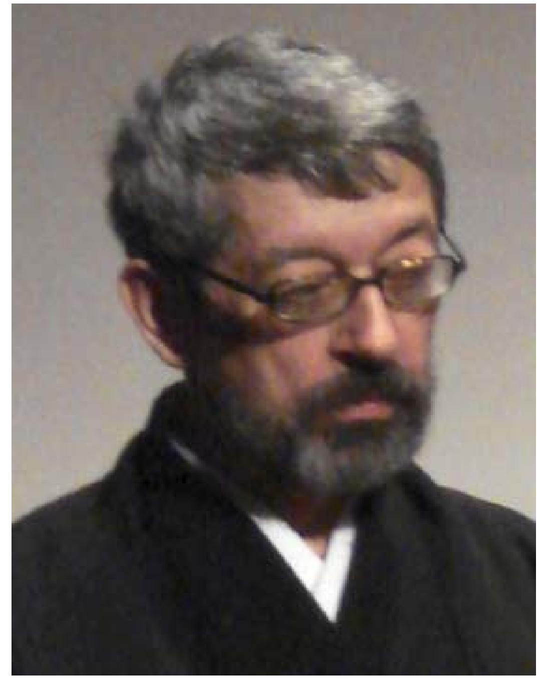
No *1-dominating set*

2-dominating set

Does *any D* always contain a *2-ds* ?

The Chvátal-Lovász Theorem (1974)

Every digraph contains a *2-dominating set*.



Chvátal (1946 –)
Canada Research Chair in Combinatorial Optimization

László Lovász (09/03/1948 –)

IMU President (2007–2010)

IMO-Gold(1964, 65, 66)

Wolf Prize(1999)

Gödel Prize(2001)

Kyoto Prize(2010)

虎父无犬子

Miklos Lovasz

IMO-Silver(2007)

& Gold(2008)

The **Jacob-Meyniel Theorem (1996)**

Every digraph which contains *no 1-dominating set* contains at least *three* *2-dominating sets*.

Landau: $k(2,T) \geq 1$

Moon: No source, $k(2,T) \geq 3$

Petrovic-Thomassen: $k(4,M) \geq 1$

Koh-Tan: No source,

$k(4,M) \geq \{3,4\}$

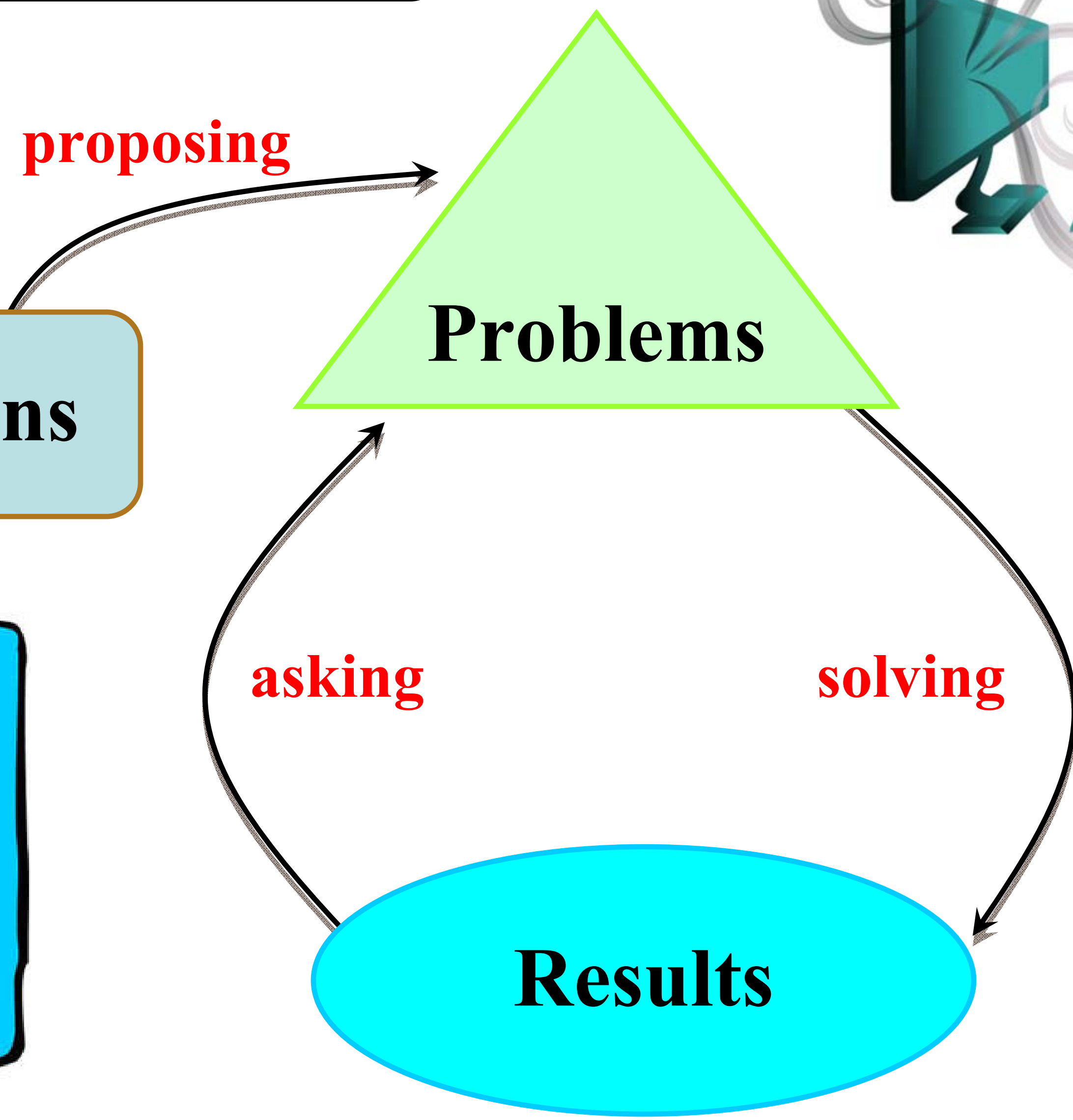
Chvátal-Lovász: $\#(2\text{-}ds,D) \geq 1$

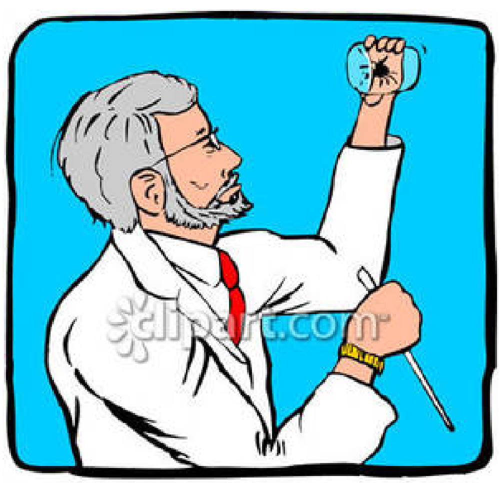
Jacob-Meyniel: No 1-ds,

$\#(2\text{-}ds,D) \geq 3$

Research Process

Observations





Art of *Problem Posing*

George Ferdinand Ludwig Philipp **Cantor**

(1845 – 1918)

Cantor founded ***set theory*** and introduced the concept of infinite numbers with his discovery of cardinal numbers

"In mathematics, the art of proposing a question must be held of higher value than solving it."

♥ **Cultivate the *Habit* of *Problem-Posing***