PROBLEMS

Only one reader (a Junior Member) submitted solutions to some of the problems in the last issue, Vol. 1, No. 1. We hope that more readers would submit (within three months) their solutions to the present set of problems. We again appeal to readers for interesting problems. If a solution to a submitted problem is known, please include the solution. Problems or solutions should be sent to Dr. Y.K. Leong, Department of Mathematics, University of Singapore, Singapore 10.

P1/74. Prove that the expressions $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of integral values of x and y. (Eötvos Competition 1894)

P2/74. Let r be a given positive odd integer. Suppose that 2^u is the highest power of 2 dividing $r + 1$, and that 2^v is the highest power of 2 dividing $r - 1$. Show that $2^{u+v+n-1}$ is the highest power of 2 dividing $r^{2^n} - 1$, where n is a positive integer. (H. N. Ng)

P3/74. A visitor is travelling in a land where each inhabitant either always tell the truth or always tell the lie. The traveller reaches a fork road, one branch of which leads to the capital while the other leads to a region of quicksand. He meets a native at the junction of the fork road. How can the traveller by asking the native only one question find the way to the capital with certainty?

(via C.T. Chong)

P4/74. Let $p(x) = a_0 + a_1x + \ldots + a_mx^m$,

$$q(x) = b_0 + b_1x + \ldots + b_nx^n,$$

be polynomials with real coefficients a_i, b_i. Suppose that r and s are two coprime positive integers such that $p(x)^r = q(x)^r$ and $p(x)^s = q(x)^s$. Show that $p(x) = q(x)$. (T.A. Peng)
(Note. The result is still true if $p(x)$ and $q(x)$ are replaced by elements in an integral domain.)

P5/74. Let n ($n \geqslant 4$) distinct points be chosen on a given circle. Join each pair of points by a straight line. Show that the maximum number of points of intersection which lie inside the circle is $n(n-1)(n-2)(n-3)/24$.

(via Y.K. Leong)

Solutions of P1 to P5/73.

P1/73(i).

Solution by Tan Nguan Sen

If x and y are positive and unequal, prove that

$$\frac{x}{2y+x} + \frac{y}{2x+y} > \frac{2}{3}.$$

Since $x \neq y$, $(x-y)^2 > 0$. Expanding and rewriting, we get

$$3(2x^2 + 2xy + 2y^2) > 2(2x^2 + 5xy + 2y^2),$$

or $$3(x(2x+y) + y(2y+x)) > 2(2x+y)(2y+x).$$

Since x and y are positive, dividing by $3(2x+y)(2y+x)$ gives the required inequality.

P1/73(ii).

With the hypothesis in (i), prove that

$$\frac{x}{y+2x} + \frac{y}{x+2y} < \frac{2}{3}.$$

Rewrite the inequality $(x-y)^2 > 0$ in the form

$$3(x^2 + 4xy + y^2) < 2(2x^2 + 5xy + 2y^2),$$

or $$3(x(x+2y) + y(y+2x)) < 2(y+2x)(x+2y).$$

We now divide by $3(y+2x)(x+2y)$.

P2/73.

Solution by Tan Nguan Sen

Find the largest number that can be obtained as a product of two positive integers whose sum is a given positive integer s.

Let a and b be two positive integers, $p = ab$, and $s = a + b$. Then $4p = s^2 - (a-b)^2$. Since s is given, p is greatest when $a = b$ and the greatest value of p is $s^2/4$.

P3/73.

Solution by Proposer.

Let $m_1, m_2, \ldots$ be a sequence of positive integers. Must $(m_1^2 + \ldots m_n^2)/(m_1 + \ldots + m_n)^2$ necessarily converge to zero as n tends to infinity? If not, what is a necessary and sufficient condition for it to converge to zero? Can this problem be generalized?

Write $s_n = (m_1^2 + \ldots + m_n^2)/(m_1 + \ldots + m_n)^2$. If we take $m_i = 2^{i-1}$, then $s_n = (1 - 4^{-n})/(3(1 - 2^{-n})^2)$, which converges to 1/3 as n tends to infinity.

Write $b_{ni} = m_i/(m_1 + \ldots + m_n)$, $i = 1, \ldots, n$. Let $a_n = \max(b_{n1}, b_{n2}, \ldots, b_{nn})$. Then a necessary and sufficient condition for s_n to converge to zero as n tends to infinity is that a_n converges to zero as n tends to infinity. This follows from the inequalities

$$a_n^2 \leq s_n \leq a_n ,$$

which are easily deduced from the fact that

$$b_{ni}^2 \leq b_{ni} , \quad s_n = b_{n1}^2 + \ldots + b_{nn}^2 .$$

The above result can be generalized as follows. A necessary and sufficient condition for $(m_1^k + \ldots + m_n^k)/(m_1 + \ldots + m_n)^k$, $k > 1$, to converge to zero as n tends to infinity is that a_n converges to zero as n tends to infinity, where a_n is defined as before. The proof is similar.

P4/73.

Prove $\displaystyle\int_0^x \frac{t^n}{1+t}\,dt = \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt$.

Introduce a change of variables $t = (x - u)/(1 + u)$ in the integral on the left hand side.

Let p(x) be a polynomial with integral coefficients, and let x_1 be an even integer and x_2 an odd integer. If $p(x_1)$ and $p(x_2)$ are both odd, prove that p(x) = 0 has no integral roots.

Now $p(x) = a_o + a_1x + \ldots + a_nx^n$ where $p(x_1)$ and $p(x_2)$ are both odd for an even integer x_1 and an odd integer x_2. Let $x_1 = 2r$ and $x_2 = 2s + 1$. Using the binomial theorem, we see that

$$p(x_1) = a_o + 2k, \quad p(x_2) = a_o + a_1 + \ldots + a_n + 2m$$

for some integers k and m. Hence a_o and $a_1 + \ldots + a_n$ are both odd. This fact(together with an application of the binomial theorem) implies that p(x) is always odd for any integer x and so cannot be zero.

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BOOK REVIEWS

Mathematical Structures. By H. T. Combe. Ginn, London; 1971, v + 201 pp; $12.

This well-written book provides an elementary approach to a number of mathematical structures such as groups, rings, fields, [illegible] vector spaces and Boolean algebra, but the emphasis is on groups. There is also a chapter on the number systems and one on applications of Boolean algebra.

The author assumes that the reader has a knowledge of the basic set operations and the notations and compositions of functions, as well as, an acquaintance with the simple matrix operations. Since the knowledge assumed is nothing more than the basic definitions, which can be found in almost all modern secondary school textbooks, the book should be easily understood by all those who have done a few years of "Modern Mathematics" in a lower secondary school.

There are numerous exercises some of which are questions set by examining bodies in the United Kingdom. Answers to most questions are provided. Although the book will be most useful to H.S.C. students studying "Modern Mathematics",it