ON THE GENERAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

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In this article we try to justify why the general solution of the linear differential equation of second order involves two arbitrary constants.

In schools, the solutions of the differential equation of the form

$$\ddot{y}(t) + a\dot{y}(t) + by(t) = 0, \qquad (1)$$

for instance, the wave equation $\ddot{y}+\omega^2 y=0$, are usually discussed. When two special solutions $y_1(t)$, $y_2(t)$ of (1) are found, the teacher comes out with the startling statement that every solution of (1) can be expressed in the form

$$c_1y_1(t) + c_2y_2(t) , \qquad (2)$$

where c_1 and c_2 are arbitrary constants. Some teachers, presumably to dampen this shock, may add that this fact can be established by using the uniqueness theorem (stated below), linearity of (1) and concepts from the theory of "vector spaces". Even the student who was about to stand up and press the teacher for a plausible explanation of this fact will, after this "explanation involving terms that he little understands, quietly sit back mumbling "Oh, it is one of those things". The reason for this article is from the belief that the teacher can "lessen" this "uneasiness" by showing that the above fact is plausible without explicit reference to vector space and basis. The usual trick for sneaking in an algebraic concept is to use geometrical diagrams, and this is exactly what we shall do here. We think that the student can accept the following uniqueness theorem without much difficulty.

<u>Uniqueness theorem</u>* Given two constants α and β there is a unique solution of (1) satisfying the following initial conditions:

$$\left.\begin{aligned} y(t_0) &= \alpha, \\ \dot{y}(t_0) &= \beta. \end{aligned}\right\} \quad (3)$$

Let us now choose any two (non-trivial) solutions of (1) and then show that every solution of (1) is of the form (2). Let $(\alpha_1,\beta_1),(\alpha_2,\beta_2)$ be two pairs of constants such that neither pair is (0,0). Let $y_1(t)$ be the unique solution of (1) with the initial conditions

$$\left.\begin{aligned} y(t_0) &= \alpha_1 , \\ \dot{y}(t_0) &= \beta_1 \end{aligned}\right\}$$

and $y_2(t)$ be the unique solution of (1) with the initial conditions

$$\left.\begin{aligned} y(t_0) &= \alpha_2 , \\ \dot{y}(t_0) &= \beta_2 . \end{aligned}\right\}$$

Our objective is to show that every solution of (1) can be expressed in the form

$$c_1y_1(t) + c_2y_2(t)$$

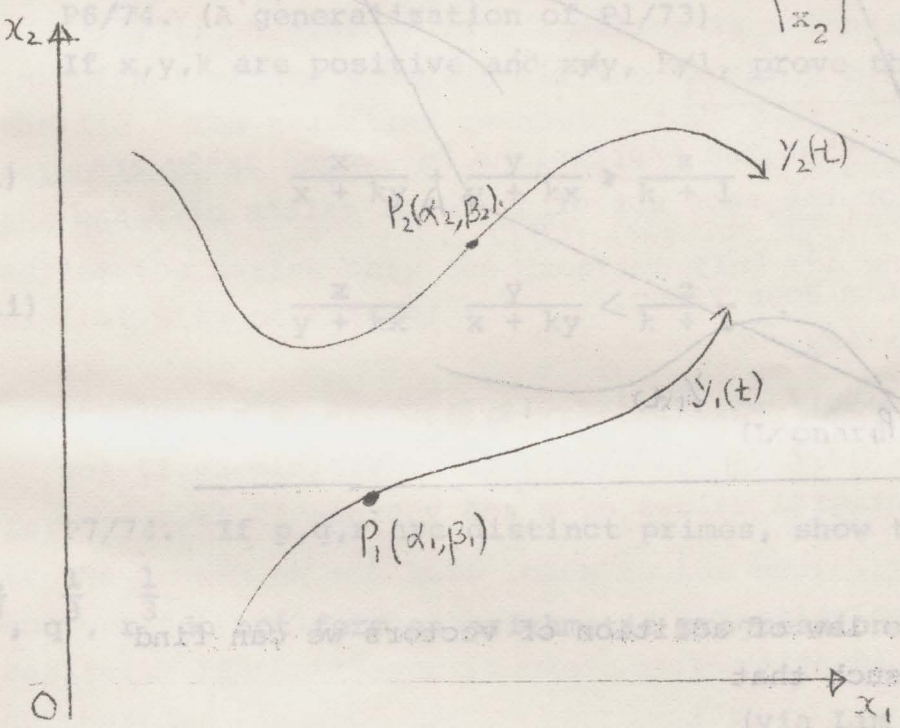
* A sketch of the proof may be found in "Introduction to ordinary differential equations", by A. L. Rabenstein, Academic Press (1966), pp. 29-31.

As a first step, we rewrite (1) as follows:

$$\left.\begin{aligned} y &= x_1 \\ \dot{y} &= x_2 \end{aligned}\right\} \Longrightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - bx_1, \end{aligned}$$

i.e. $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

i.e. $\dot{\underline{x}} = A\,\underline{x}$, where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}$



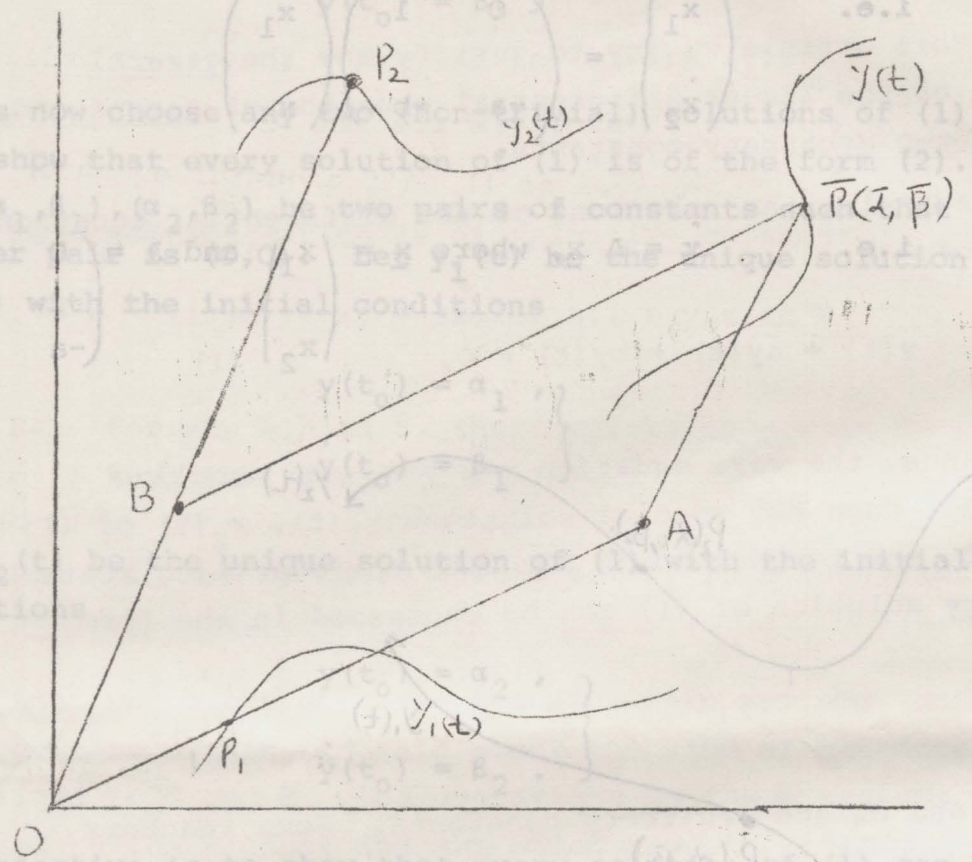
We consider the Cartesian plane with axes Ox_1x_2, sometimes called the phase space. Then by virtue of the above formulation of (1) we can view the solutions of (1) as curves in this plane represented parametrically by

$$x_1 = x_1(t)$$

$$-\infty < t < \infty$$

and $$x_2 = x_2(t)$$

Thus the solution $y_1(t)$ is a curve in this plane which passes through the point $P_1(\alpha_1, \beta_1)$ at time t_o. Now consider any solution $\bar{y}(t)$ of (1). Since this is a solution of (1), it can be viewed as a curve in this plane. Let us take it that this curve $\bar{y}(t)$ passes through the point $\bar{P}(\bar{\alpha}, \bar{\beta})$ at time t_o



Using the triangle law of addition of vectors we can find constants c_1, c_2 such that

$$\underset{\sim}{O\bar{P}} = c_1 \; \underset{\sim}{OP}_1 + c_2 \; \underset{\sim}{OP}_2$$

Now consider the function $\phi(t) = c_1 \, y_1(t) + c_2 \, y_2(t)$. Clearly this curve passes through $\bar{P}(\bar{\alpha}, \bar{\beta})$ at time t_o by virtue of our choice of c_1, c_2. Further as $y_1(t)$ and $y_2(t)$ satisfy (1) $[c_1 \, y_1(t) + c_2 \, y_2(t)]$ will also satisfy (1) for arbitrary constants c_1 and c_2. This is the linearity property of (1). Thus $\phi(t)$ and $\bar{y}(t)$ are solutions of (1) and they pass through the same point $\bar{P}(\bar{\alpha}, \beta)$ at time t_o. Hence by the uniqueness theorem

$$\bar{y}(t) = \phi(t) = c_1 y_1(t) + c_2 y_2(t)$$

which establishes the original claim.

PROBLEMS AND SOLUTIONS

Problems or solutions should be sent to Dr. Y.K.Leong, Department of Mathematics, University of Singapore, Singapore 10. If a solution to a submitted problem is known, please include the solution.

P6/74. (A generalization of P1/73)

If x,y,k are positive and $x \neq y$, $k \neq 1$, prove that

(i) $$\frac{x}{x+ky} + \frac{y}{y+kx} > \frac{2}{k+1},$$

(ii) $$\frac{x}{y+kx} + \frac{y}{x+ky} < \frac{2}{k+1}.$$

(Leonard Y. H. Yap)

P7/74. If p,q,r are distinct primes, show that $p^{\frac{1}{3}}$, $q^{\frac{1}{3}}$, $r^{\frac{1}{3}}$ do not form an arithmetic progression in any order.

(via Lim Boon Tiong)

P8/74. Without using tables, show that

$$\frac{1.3.5...99}{2.4.6...100} < \frac{1}{10}.$$

(via Louis H. Y. Chen)

P9/75. Show that for all positive integers m and n,

$$\binom{m+0}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \ldots + \binom{m+n}{n} = \binom{m+n+1}{n},$$ where