$$\bar{y}(t) = \phi(t) = c_1y_1(t) + c_2y_2(t)$$

which establishes the original claim.

PROBLEMS AND SOLUTIONS

Problems or solutions should be sent to Dr. Y.K.Leong, Department of Mathematics, University of Singapore, Singapore 10. If a solution to a submitted problem is known, please include the solution.

P6/74. (A generalization of P1/73)

If x,y,k are positive and $x \neq y$, $k \neq 1$, prove that

(i) $$\frac{x}{x+ky} + \frac{y}{y+kx} > \frac{2}{k+1},$$

(ii) $$\frac{x}{y+kx} + \frac{y}{x+ky} < \frac{2}{k+1}.$$

(Leonard Y. H. Yap)

P7/74. If p,q,r are distinct primes, show that $p^{\frac{1}{3}}$, $q^{\frac{1}{3}}$, $r^{\frac{1}{3}}$ do not form an arithmetic progression in any order.

(via Lim Boon Tiong)

P8/74. Without using tables, show that

$$\frac{1.3.5...99}{2.4.6...100} < \frac{1}{10}.$$

(via Louis H. Y. Chen)

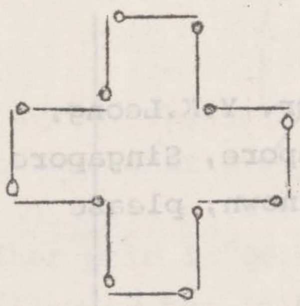
P9/75. Show that for all positive integers m and n,

$$\binom{m+0}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \ldots + \binom{m+n}{n} = \binom{m+n+1}{n}$$, where

$\binom{m+i}{i}$ is the binomial coefficient $(m+i)!/m!i!$.

(Y. K. Leong)

P10/75. Twelve matches, each of unit length, form the figure of a cross as shown. Rearrange the matches in such a way as to cover an area of four square units.



(from "Figures for fun" by Ya. Perelman)

<u>Solutions to P1 - P5/74</u>

P1/74. Prove that the expressions $2x+3y$ and $9x+5y$ are divisible by 17 for the same set of integral values of x and y.

Solution by Lim Boon Tiong.

From the identity,

$$(2x+3y)^2 + (9x+5y)^2 = 17(5x^2+6xy+2y^2),$$

it follows that $(2x+3y)$ and $(9x+5y)$ are divisible by 17 for the same set of integral values of x and y since 17 is a prime.

Alternative solution.

Consider the identity,

$$9(2x+3y) - 2(9x+5y) = 17y,$$

and the fact that 17 is coprime to both 9 and 2.

P2/74. Let r be a given positive odd integer. Suppose that 2^u is the highest power of 2 dividing $r+1$ and that 2^v is the highest power of 2 dividing $r-1$. Show that $2^{u+v+n-1}$ is the highest power of 2 dividing $r^{2^n}-1$, where n is a positive integer.

Solution by Lim Boon Tiong.

Use induction on n. The result is true for n=1 since $r^2-1 = (r-1)(r+1)$. So assume the result to be true for n $n=k \geqslant 1$. Thus from the identity

$$r^{2^{k+1}} - 1 = (r^{2^k} - 1)(r^{2^k} + 1),$$

it remains to show that 2^1 is the highest power of 2 dividing $r^{2^k} + 1$. Since r is odd, $r^{2^{k-1}} = 2s+1$ for some integer s, and so

$$r^{2^k} + 1 = (2s+1)^2 + 1 = 2(2s^2+2s+1).$$

Hence the result is also true for n=k+1.

P3/74. A visitor is travelling in a land where each inhabitant either always tells the truth or always tells the lie. The traveller reaches a fork road, one branch of which leads to a region of quicksand. He meets a native at the junction of the fork road. How can the traveller by asking the native only one question find the way to the capital with certainty?

Solution by Proposer.

One such question the traveller could ask the native is: "Is it true that you always tell the truth if and only if the branch on the left leads to the capital?" If the answer is "yes", then the branch on the left leads to the capital. If the answer is "no", then it leads to the quicksand.

To see this, let A be the proposition "You (i.e. the native) always tell the truth" and B the proposition "The branch on the left leads to the capital." Set up the following truth table for the proposition $A \Longleftrightarrow B$:

A	B	$A \Longrightarrow B$	$A \Longleftarrow B$	$A \Longleftrightarrow B$
T	T	T	T	T
F	T	T	F	F
F	F	T	T	T
T	F	F	T	F

where "T" and "F" denote "true" and "false" respectively. From this table, we see that if A is true (i.e. the native always tells the truth), then the truth values of $A \Longleftrightarrow B$ coincide with those of B. On the other hand, if A is false (i.e. the native always tells the lie), then the truth values of $A \Longleftrightarrow B$ are the opposite of those of B. But being what he is, this native will reverse the truth values of $A \Longleftrightarrow B$ in his answer. The effect is that the native's answer will correspond to the truth value of B no matter who gives the answer to the question posed.

P4/74. Let $p(x) = a_0+a_1x+\ldots+a_mx^m$, $q(x)=b_0+b_1x+\ldots+b_nx^n$ be polynomials with real coefficients a_i, b_i. Suppose that r and s are two coprime positive integers such that $p(x)^r = q(x)^r$ and $p(x)^s = q(x)^s$. Show that $p(x) = q(x)$.

Solution.

There are positive integers k and ℓ such that $rk-s\ell=1$. Hence $p(x)^{rk} = p(x).p(x)^{s\ell}$, or $q(x)^{rk} = p(x)q(x)^{s\ell}$. But $q(x)^{rk} = q(x).q(x)^{s\ell}$. Thus

$$(p(x) - q(x))\, q(x)^{s\ell} = 0.$$

We may assume that q(x) is not the zero polynomial. Hence $p(x) - q(x) = 0$.

P5/74. Let $n(n \geqslant 4)$ distinct points be chosen on a given circle. Join each pair of points by a straight line. Show that the maximum number of points of intersection which lie inside the circle is $n(n-1)(n-2)(n-3)/24$.

Solution by Tan Nguan Sen

The number of points of intersection which lie inside the circle is maximum when no three lines are concurrent. Each required point of intersection arises from 4 distinct points on the circle; it is the intersection of the diagonals of the quadrilateral with the 4 points as vertices. Hence the required number is just the number of quadrilaterals that can be formed from the n given points, namely $n(n-1)(n-2)(n-3)/24$.

Correction to the solution of P2/73 (this "Medley" Vol.2, No.1, p.20).

Find the largest number that can be obtained as a product of two positive integers whose sum is a given positive integer s.

The solution as presented is only a partial solution (when s is even) as pointed out by Dr. Louis Chen and the Proposer. The complete solution is as follows.

Let $p = ab$, $s = a+b$ where a and b are positive integers. Then $4p = s^2 - (a-b)^2$. Since s is given p is greatest when $(a-b)^2$ is smallest. If s is even, then we can choose a to be equal to b. If s is odd, then a and b are of different parity and so $(a-b)^2$ is at least 1. Hence the largest value of p is $s^2/4$ or $(s^2-1)/4$ according as s is even or odd.

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BOOK REVIEWS

Books for review should be sent to Dr. L. Y. Lam, Department of Mathematics, University of Singapore.

An Introduction to Sets. By A.P. Kearney. Blackie, London, Glasgow, 1965, 159 pp. 85 pence.

Sets and Switches: Using the Karnaugh Map Technique". By Alan Sherlock and Timothy Brand. Chatto & Windus Educational, London, 1973, 117 pp. 70 pence.

In Kearney's book, the concepts of sets, equivalent sets, operations on sets (intersection, union, difference and complementation) are introduced within some 45 pages with the help of numerous Venn diagrams and illustrative examples. The author also mentions briefly the real number system: the rational numbers and irrational numbers, and in particular, the integers. Using the new terminology, he describes the graphs of inequalities, relations and functions.

In the latter part of the book, the author attempts to present abstract proofs of set identities. He also mentions the alternative method of proof by "membership" (or "in-out") tables. Finally there are some brief remarks about the laws of Boolean algebra.