WHAT IS MODERN ALGEBRA?

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The term modern algebra was introduced in the 1930's to distinguish it from classical algebra which was generally understood to mean the theory of equations. According to an eminent American mathematician (G. Birkhoff), the theory of equations may be defined as "the art of solving numerical problems by manipulating symbols". The central idea of modern algebra is that of an algebraic system by which we mean a set of objects (called elements) together with one or more rules of combination (called operations) of these elements. The most common rule of combination is a binary operation. This is a rule which assigns to each ordered pair of elements of the set another element of the set. The set of all integers together with the operations of addition and multiplication is an algebraic system. But, in general, the elements of an algebraic system are not numerical and the operations have to be defined. In defining the operations we have to lay down a number of assumptions (called postulates or axioms) on the behaviour of the elements with respect to the operations. Once the axioms are laid down we can begin to deduce consequences (called theorems) using logical arguments.

This makes modern algebra sound like a game. But there are games and games. Some games have many players and some have only a devoted few. In general, algebraists do not write down axioms in order to play games. Most of the algebraic systems have ultimately something to do with other branches of mathematics, which may or may not be applicable outside mathematics. As far as we know, most branches of mathematics have some applications in the real world. When complex numbers were first studied, it was because they were useful in solving equations and not because they had any application in the real world. Nowadays, one can hardly imagine the study of aerodynamics or electronics without these "imaginary numbers".

Before we take a closer look at one or two of such systems, let us have a brief survey of some of the developments in algebra in the eighteenth and nineteenth centuries.

Some highlights of classical algebra

1750-1830

During this period the Fundamental Theorem of Algebra was actively studied. This theorem states that every polynomial equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0 ,$$

where n is a positive integer and $a_0, a_1, \ldots, a_n$ are complex numbers, has a complex root.

By 1770, J. L. Lagrange (1736-1813) already realised that the symmetric group had some relevance to the solution of a polynomial equation by radicals.

L. Euler (1707-1783) considered the real form of the Fundamental Theorem of Algebra but his proof is obscure.

C. F. Gauss (1777-1855) was the first to give a number of rigorous proofs of the Fundamental Theorem of Algebra from 1800 onwards. Gauss also developed systematic iterative and elimination techniques for solving simultaneous equations in many unknowns.

The search for general solutions by radicals of polynomial equations of degree greater than or equal to 5 turned out to be futile when N. H. Abel (1802-1829) and E. Galois (1811-1832) proved that such solutions do not exist.

1830-1860.

Galois and A. Cauchy (1789-1857) made significant contributions to the development of group theory before 1845. In 1830 Galois constructed a finite field of each prime power order. (A field is an algebraic system similar to the set of rational numbers with the operations of addition and multiplication. A finite field is one which has a finite number of elements.).

Gauss and Legendre (1752-1833) initiated the study of commutative rings, (which are systems similar to the set of integers under addition and multiplication).

Non-commutative rings, vector algebra and matrix algebra were introduced and studied by W. Hamilton (1805-1865), H. Grassmann (1809-1877) and A. Cayley (1821-1895). Matrix algebra eventually developed into what is known today as linear algebra.

G. Boole (1815-1864) published in 1854 his "Introduction to the Laws of Thought". This is the origin of Boolean algebra.

1860-1914

During this period the axiomatic approach of modern algebra was developed.

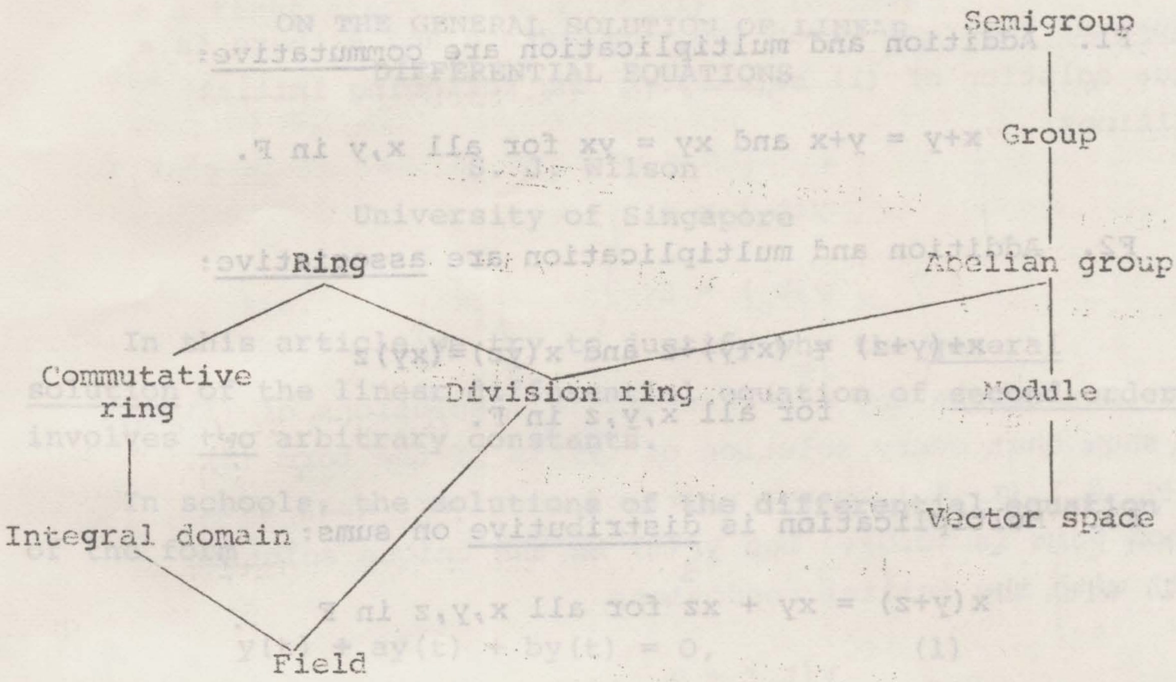
The axiomatic treatment of groups, fields and rings was studied by Cayley, G. Frobenius (1849-1917), R. Dedekind (1821-1917) and B. Peirce (1809-1880).

In 1888, G. Peano (1858-1932) initiated the axiomatic approach to arithmetic. This was developed by B. Russell (1872-1970) and A.N. Whitehead (1861-1947). In 1899, D. Hilbert (1862-1943) in "Foundations of Geometry" initiated the study of axiom systems.

Dedekind also proved the unique factorization of algebraic number fields. The study of algebraic number fields is to-day one of the most active branches of algebra.

The above brief survey shows that most of what we call modern algebra was known before 1914. Modern algebra in its present form is due mainly to mathematicians like E. Noether, E. Artin and B.L. van der Waerden. In particular the book "Moderne Algebra" by van der Waerden published in 1931 not only made modern algebra respectable but also popular. If we are to choose one book which has the greatest impact on algebra in the twentieth century, this must be it. But by 1959 the material in van der Waerden's book was no longer modern and he changed the title of the book to "Algebra". So what was known as modern algebra in the 1930's has become almost classical by 1960

The following diagram illustrates the relationships between some major algebraic systems which are being actively pursued.



We now select two algebraic systems and have a closer look at their axioms.

The first system is a group. The following axioms were given by E. V. Huntington in 1906.

<u>Definition</u> A <u>group</u> is a set G of elements (to be denoted by small Latin letters), any two of which, say x and y, have a <u>product</u> xy which satisfies the following conditions:

G1. Multiplication is <u>associative</u>: $x(yz) = (xy)z$ for all x,y,z in G.

G2. For any two elements a,b in G, there exist x,y in G such that $xa = b$ and $ay = b$.

From these axioms one can deduce, by using general principles of logic, various simple conditions which can serve as alternative definitions of a group. These can be found in any book on modern algebra.

The second algebraic system is a field. The following set of axioms is also due to Huntington:

<u>Definition.</u> A <u>field</u> is a set F of elements, any two of which have a <u>sum</u> x+y and a <u>product</u> xy which satisfy the following conditions:

F1. Addition and multiplication are commutative:

$$x+y = y+x \text{ and } xy = yx \text{ for all } x,y \text{ in } F.$$

F2. Addition and multiplication are associative:

$$x+(y+z) = (x+y)+z \text{ and } x(yz)=(xy)z$$
$$\text{for all } x,y,z \text{ in } F.$$

F3. Multiplication is distributive on sums:

$$x(y+z) = xy + xz \text{ for all } x,y,z \text{ in } F$$

F4. For any a,b in F, there exists some x in F such that $a+x=b$.

F5. If $a+a \neq a$, there exists some y in F such that $ay = b$.

When one looks at these two algebraic systems, one notices that nothing is said about the kind of elements contained in G and F. The operations in these systems are defined by the product and sum which have to satisfy certain conditions. These terms are borrowed from ordinary arithmetic so that they may be more easily understood though any other terms will do. The actual operations in a specific group or field may be of importance, but in the general study of these systems we do not have to worry about them at all. All we need to know are the conditions which they satisfy. Whether we can deduce any significant results from the axioms of an algebraic system is another matter. The fact that we are not shackled by any particular system such as that of the real or complex numbers is one of the main reasons for the flowering of algebra in the twentieth century.