Some Suggestions for further reading

J. N. Crossley and Others: What is Mathematical Logic? Oxford University Press 1972 (paperback)

H. Enderton: A Mathematical Introduction to Logic Academic Press 1972.

S. C. Kleene: Mathematical Logic. Wiley 1967.

R. C. Lyndon: Notes on Logic, van Nostrand lecture notes (paperback). 1966.

E. Mendelson: Introduction to Mathematical Logic, van Nostrand 1964.

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NOTES ON MATHEMATICIANS

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Introduction

Pure mathematics splits fourfold into (1) Mathematical Logic and the Foundations of Mathematics, (2) Algebra and the Theory of Numbers, (3) Analysis and Geometry and (4) Topology. Of these subjects, the Theory of Numbers and Geometry undoubtedly have the longest history, dating back to some two thousand years. Other subjects developed much later, with Analysis and Algebra around the same time in the eighteenth century, Topology in the late nineteenth century, and finally Logic at the turn of this century.

Over the years these subjects evolve and intermingle with one another, while at the same time they expand further and further at a fascinating rate. Today there is hardly any mathematician who can claim to be a universalist. Not so fifty years ago! David Hilbert eminently qualified as one, his contributions having covered every subject in pure and applied mathematics. Before him there were Poincaré and Gauss. But still there have been very few.

How did the subjects evolve? Who were the prime movers, the great contributors? In this series of notes we shall introduce men who in our opinion develop mathematics into what it is today, and because of whom mathematics is never the same again.

A great mathematical work is like a great work of art. It is the result of a complete devotion to the subject, a highest degree of concentration of the mind and an exploitation to the limit of man's faculty of thought, by which intricate and ingenious logical arguments are conceived to take care of alll the difficulties involved in the successful completion of a work.

Therefore it takes years to accomplish such a feat. The satisfaction lies in seeing a rock gradually getting carved into perfect shape in front of your eyes to become a great work of art, a work that will go down in history. Each of the mathematicians that we will take up in this series is one of such calibre — a creator and pioneer of knowledge whose great achievements few in the history of mankind could ever attain.

1. Carl Friedrich Gauss (1777–1855)

It is often said that great men leave their marks behind when they are gone. G. H. Hardy [1] said that great mathematicians produced their greatest works before the age of forty (which, perhaps by coincidence, is also the age limit set by the awarding of the Fields Medal [2]). It is generally agreed in the mathematical circle that a child who shows exceptional talent in mathematics at a tender age has, in most cases, a great potential to become a research mathematician. Carl Friedrich Gauss met and surpassed each of these requirements. To E.T. Bell [3] , he was "the Prince of Mathematicians" . To his contemporary, Laplace [4] , he was "the greatest mathematician of the world" (and he was then only about 30!). The name of Gauss is knit to almost all of the major fields of mathematics of our time. Among the great masters of this discipline, he belongs to an exclusive class joined only by a select few.

He was born in Brunswick, Germany, on the last day of April. The parents were poor and his father never cared much for education and tried very hard to make him become a gardener. Fortunately, his uncle found him to be precocious and tried all that he could to stimulate the boy's enquiring mind.

The first sign of Gauss' originality in tackling mathematical problems came when he was but ten years old. A series of numbers were given to the class and the students were asked to calculate the sum of these numbers. The numbers were arranged in a sequence which we know today in schools as forming an arithmetical progression. The formula for finding such a sum had not been taught. Neither were the students told of any possible way of solving the problem. Nevertheless, Gauss at once visualized an underlying pattern among the numbers, and using it, wrote down the answer. To the end of his days, he loved to tell how the one number he found was the only correct answer in the class.

At the age of nineteen, he proved the famous "Law of Quadratic Reciprocity". It may be stated as follows:

> Let p,q be any two distinct primes. Write $(\frac{p}{q}) = 1$ if there is an integer x such that x^2-p **is divisible by q,** and write $(\frac{p}{q}) = -1$ otherwise. Define $(\frac{q}{p})$ in a similar manner. Then $(\frac{p}{q}) = (\frac{q}{p})(-1)^k$, where $k = \frac{p-1}{2} \cdot \frac{q-1}{2}$.

The interested reader might want to try to prove this statement. It is a difficult one and earlier mathematicians like Euler [5] and Legendre [6] were baffled by it. This Reciprocity Law is of fundamental importance in higher Arithmetic (better known as the Theory of Numbers). Gauss turned it over and over in his mind for many years until in the end he had given six different proofs. The Reciprocity Law continued to attract the attention of mathematicians after Gauss' time. By the 1930's there had been 56 known proofs, one of which was given by Cauchy [7] and was somehow related to heat conduction!

Gauss had entered the University of Göttingen a year before he proved the Reciprocity Law. (The University was, for the next hundred and fifty years, to remain as one of the mathematical centers of the world. In later **articles**, we shall have occasion to come across mathematical activities associated with it). The three years (1795-1798) he spent there were the most prolific in his life. He entered the University still undecided whether to take up mathematics or philology as his lifework. By the time he left the place it was mathematics for him, and his great work, Disquisitiones Arithmeticae(Arithmetical Researches), was near completion. The Disquisitiones (published in 1801) is certainly a monumental work whose contributions to the Theory of Numbers equal those of Euclid's Elements to Geometry. It brought forth a new direction to the higher arithmetic. The subject which previously had been a collection of isolated results, now assumed coherence and rose to the dignity of a mathematical science. The work was received with great enthusiasm. Legendre wrote to Gauss in 1804, saying, "Your Disquisitiones has raised you at once to the rank of the first mathematicians ... Believe, sir, that no one applauds your success more sincerely than I." Gauss was then 27.

After the appearance of the Disquisitiones, he broadened his research activity to include astronomy, geodesy, and electromagnetism in both their mathematical and practical aspects. But Arithmetic was his first love, and he regretted in later life that he had never found the time to write the second volume he had planned as a young man. In any case, his computations of the planetary orbits of Ceres and Pallas brought him even greater fame, established him as the foremost mathematician of Europe, and won him the Directorship of the Göttingen Observatory.

He got married at 28 and four years later became a widower with three children. Although he married again the following year for the sake of his children, it was long before Gauss could speak without emotion of his first wife.

In 1808 he lost his father. Two years previously he had suffered an even more severe loss in the death of his benefactor, the Duke of Ferdinand. As was mentioned earlier, Gauss had come from a poor family. However, owing to the generosity of the Duke, the young man did not have to worry about finances during his three years of study at Göttingen. Later in 1799, when he was short of money for the publication of his doctoral thesis, the Duke also came to his rescue. The financial support he received during this period enabled him thus to concentrate on his work. It was therefore not surprising that Gauss dedicated the Disquisitiones to the Duke. Indeed one wonders whether he could have contributed so much to mathematics were financial difficulties to deny him the opportunity to create.

It would take too much space to describe all the outstanding scientific and mathematical works of Gauss. We shall mention only a few here. He laid the foundations of the theory of electromagnetism and invented the electric telegraph. He introduced the method of least squares which is now contained in every standard textbook of statistics. In his second masterpiece (after the Disquisitiones)-- Theory of the Heavenly Bodies Revolving Around the Sun in Conic Sections -- he laid down the law which for many years was to dominate computational and practical astronomy. Turning to the mathematical side, he discovered the Theory of Functions of a Complex Variable and proved what we know today as its fundamental theorem (otherwise known as Cauchy's Theorem). But this result he hid away to be rediscovered later by Cauchy and others, who made the theory one of the greatest triumphs in 19th century mathematics. His contributions to higher Arithmetic had been mentioned earlier, but one thing deserves special note: he never attempted Fermat's Last Theorem [8] despite the urging of his friends and the prize offered by the Paris Academy in 1816 for its solution. In one of the letters that he wrote to his friends, he mentioned that he had envisaged an extension of the higher arithmetic which, if the principal steps of that new theory were successfully taken, would make Fermat's Theorem one of the corollaries. It is understood today that the theory he referred to is what we call Algebraic Number Theory, to which Kummer, Dedekind, Kronecker, Hilbert, Artin, and (in our generation) Weil, Chevalley, Serre, Tate [9] and many others, were to make significant contributions. Gauss never took time out to look into the theory as he was then deeply involved with mathematical astronomy. Probably all mathematicians today regret that Gauss did not go back to the work he did best. And of course, Fermat's Theorem remains unsolved today. It is worth noting that a large part of the modern Theory of Algebraic Numbers stemmed from attempts to solve this problem.

In the field of Geometry, Gauss was the first to envisage the possibility of a non-Euclidean Geometry, results of which he again refrained from publishing. He was also the first to have investigated in its generality geometry on curved surfaces, and from this the first great period of Differential Geometry developed. One of the fundamental theorems in this field still bears his name today: the Gauss-Bonnet Formula.

Finally there was a passing mention of Topology in his doctoral thesis, and he predicted that it would be one of the chief concerns of mathematics. The subject was later taken up by Poincaré, and notably by a group of great mathematicians at Princeton University, U.S.A. (with which names like Alexander, Lefschetz, Veblen, Steenrod, Milnor and so on are associated). Today Topology has far-reaching ramifications in both Geometry and Analysis.

How was it possible for one man to accomplish this colossal mass of work of the highest order? With characteristic modesty Gauss declared that "if others would but reflect on mathematical truths as deeply and as continuously as I have, they would make my discoveries." And this takes devotion, concentration, and fortitude, but most of all, love for mathematics. For Gauss, there was more: his involuntary preoccupation with mathematical ideas. Conversing with friends he would suddenly go silent, overwhelmed by thoughts beyond his control, and stood rigidly oblivious of his surroundings. A problem once grasped was never released till he had conquered it. Indeed there was one instance where for four years hardly a week passed that he did not spend some time trying to settle whether a certain sign should be plus or minus.

Although the latter half of his life was crowned with honour, he continued to lead a simple life at Göttingen. Till his last illness he found complete satisfaction in his science and his recreations. He read widely, from classics of antiquity to books on politics. Apart from German, he was an expert in French, Latin, English and Russian which he acquired when he was sixty. He had no personal ambition. All his ambition was for the advancement of mathematics. Göttingon became the mathematical Mecca of the time, and Gauss did everything to help train young men who went there to learn mathematics. Among them was another great mathematician: Riemann [10].

In early 1855 Gauss fell ill. He suffered greatly from an enlarged heart and shortness of breath. Nevertheless he worked when he could. Early on the morning of February 23, he died peacefully. He was seventy-eight.

And the work which he left behind was taken up by others.

Notes

[1] G. H. Hardy (1877-1947), one of the greatest mathematicians of his time, taught at both Cambridge and Oxford Universities in England. His most original work was done in the analytic theory of numbers and allied subjects.

[2] The Fields Medal is awarded every four years at the meeting of the International Congress of Mathematicians to those mathematicians under the age of forty who are adjudged by committee of distinguished mathematicians to have accomplished great mathematical works.

[3] E. T. Bell (1883-1960) taught at the California Institute of Technology, U.S.A. His research in analytic number theory won him the Bôcher Prize awarded by the American Mathematical Society in 1924. His classic "Men of Mathematics" remains a popular book on the history of mathematics.

[4] Laplace (1749-1827), French mathematician at the time of Napoleon.

[5] Euler (1707-1783), Swiss mathematician.

[6] Legendre, French mathematician contemporary with Gauss.

[7] Cauchy (1789-1857), French mathematician who was one of the pioneers of the Theory of Functions of a Complex Variable.

[8] Fermat (1601-1665), French mathematician. He claimed to have a proof of the following statement: For all integers $n > 2$, $x^n + y^n = z^n$ has no integral solutions. However, a proof was not supplied and till this day the Last Problem remains one of the most well-known open problems in mathematics.

[9] Kummer, Dedekind, Kronecker, German mathematicians who lived in the period 1810-1910.
Hilbert (1862-1943), Artin(1898-1962), German mathematicians who taught at Göttingen and Hamburg Universities respectively (Artin later taught at Princeton University, U.S.A. for some time).
Chevalley, Serre, Weil, contemporary French mathematicians.
Tate, the most outstanding student of Artin at Princeton, now professor at Harvard University, U.S.A.

[10] Riemann (1826-1866), the second mathematician to succeed Gauss' post at Göttingen. During his short life-span of 39 years, "he touched nothing in mathematics that he did not in some measure revolutionize", in the words of E.T. Bell.

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TWO VIEWS

Fontenelle: "Mathematicians are like lovers... Grant a mathematician the least principle, and he will draw from it a consequence which you must also grant him, and from this consequence another."

Goethe: "Mathematicians are like Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different."