NOTES ON MATHEMATICIANS
7. EMMY NOETHER (1882 — 1935)

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Extracted from a letter written supposedly by Albert Einstein* to the editor of the "New York Times", May 3, 1935, on the death of Emmy Noether is the following:

"The efforts of most human beings are consumed in the struggle for their daily bread, ..., but there is, fortunately, a minority composed of those who recognise early in their lives that the most beautiful and satisfying experience open to humankind are not derived from the outside, but are bound up with the development of the individual's own feeling, thinking and acting. The genuine artists, investigators and thinkers have always been persons of this kind. However conspicuously the life of these individuals runs its course, nonetheless the fruits of their endeavours are the most valuable contributions which one generation can make to its successors .... In the judgement of the most competent living methematicians, Fraulein Noether was the most significant creative mathematical genius the female sex thus far produced ...... "

Forty-five years have lapsed since the death of Emmy Noether. Still, the present living mathematicians would without hesitation share the same view as Einstein. Emmy Noether remains, up to today, the greatest of all female mathematicians; and above all, one of the most significant contributors to the development of modern algebra.

The first eighteen years. Emmy Noether was born on 23 March 1882 in Erlangen, a small university town in South Germany. Her father, Max Noether, was then a professor in Mathermatics at the Erlangen University for several years. Max Noether was well-known for his work in the field of algebraic geometry, and was considered to be one of the guiding spirits of nineteenth century algebraic geometry. In the history of mathematics it is remarkably occasional to find a mathematically talented child born in an already mathematically distinguished family. In the case of the Noethers, it was the daughter who was going to outshine the father.

Together with three younger brothers, Emmy was brought up in a family in which recognition of intellectual culture and spiritual values prevailed. One brother, Fritz, had become a distinguished physicist. The Noether family led a smooth and tranquil life in Erlangen, where the father finally retired as emeritus professor.

Emmy grew up as an ordinary girl at home, having to help with household chore. Besides, she took up clavier and dancing lessons. She was acutely short-sighted since her childhood, not attractive in her outward appearance, and not exceptional in any way, as remembered by her school teachers. She went to school in Erlangen and passed out as a qualified language teacher in French and English at the age of eighteen.

Career and Prejudice In those years girls could not be admitted as regular students to the universities in Germany. Fortunately this did not mean that the door to higher education was closed to Emmy. Female students could actually be admitted as non-matriculated students and be granted permission to sit for an examination for course credit by the professor teaching the course. And they would ultimately be allowed to take the final examination for a university certificate irrespective of whether they had obtained the prerequisite course credits. Such was the educational system in those days when prejudice against women was still accepted as part of the social practice. Later Emmy was to encounter the same obstacle in her academic career.

* The letter was inspired, if not written, by Hermann Weyl, for Einstein had never met Noether [2].

So, as a non-matriculated student, Emmy took up foreign languages and mathematics at Erlangen University between 1900 and 1902. She must have subsequently discovered her own mathematical gift and so prepared herself for the final examination in mathematics which she passed in 1903.

Like many other distinguished mathematicians at that time, Emmy was to spend a long period of her life at Goettingen University — then the mathematical Mecca of the world. Her connection with Goettingen began in 1903 when she had, as a visiting student, attended classes of eminent mathematicians like Felix Klein, Otto Blumenthal, David Hilbert and Hermann Minkowski.

In autumn the following year Emmy went back to Erlangen, as she was at last permitted to matriculate as a regular student there. She worked on her doctoral thesis under the supervision of Paul Gordan, her father's colleague as well as a close family friend. Gordan was a convinced formalist, known then as the "king of invariants". It is said that in all his papers he himself wrote the formulas only, the text being added by his friends. There exist papers of his where twenty pages of formulas are not interrupted by a single text word! Hilbert's solution of the famous Gordan's Problem [7] was regarded by Gordan as "not mathematics, by theology".

Emmy submitted her doctoral thesis on the study of invariant forms in 1907. While Gordan had doubtlessly exerted a great influence on her doctoral thesis, which is an extreme example of formal computation, Ernst Fischer, a successor of Gordan, was certainly the one who first introduced "new algebra" to her. It is indeed ironical that a student of Gordan would develop her own mathematical taste in an entirely different direction later, towards the Hilbert School of mathematical thought.

At the invitation of Hilbert, Emmy came to Goettingen in 1916. At that time, Klein and Hilbert were deeply involved in the general theory of relatively. Emmy came at the right time with her knowledge of the theory of invariants. In this respect her contribution includes an elegant mathematical formulation involving the conservation of energy and momentum with respect to transformations of the four world co-ordinates and the reduction of the problem of differential invariants to a purely algebraic one by using "normal co-ordinates".

Tradition, prejudice against women and various external circumstances prevented Emmy from ever holding an academic position commensurate with her mathematical merits. At Erlangen she had lectured occasionally in place of her father when the latter fell sick. In an attempt to support her application for "Habilitation" (a licence to teach at a German University) in Goettingen, Hilbert once remarked at the Philosophical Faculty meeting, "I do not see that the sex of the candidate is an argument against her admission as Privatdozent. After all, we are an university establishment and not a bathing establishment." He, however, lost the battle against the philologists and historians. Anyway, Emmy was able to deliver lectures that were announced under Hilbert's name.

It was only in 1919, after the War, that her Habilitation was finally granted. Three years later she was nominated as a "nicht-beamteter ausserordentlicher Professor" (non-government extraordinary professor) carrying no salary and no obligation. Thereafter she was offered a "Lehr auftrag" (a teaching appointment) in algebra, through which she earned a modest remuneration.

Emmy spent the winter of 1928-1929 in Moscow, where she taught a course in abstract algebra and conducted a seminar in algebraic geometry. She established contact with a number of Russian mathematicians, including L. S. Pontnjagin and O. U. Schmidt. Already she had made her acquaintance with P. S. Alexandroff and P. S. Urysohn during their earlier visits to Goettingen. She was said to show great interest in the mathematical development and scientific situation in the Soviet Union. At one time she was accused of being "Maxist-inclined" and was banished from one of the Goettingen boarding houses. In fact in the early days, after the Revolution of 1918, Emmy did take an active part in political and social discussions in Goettingen, and sided with the Social Democrats. That was the time when young people in Germany were trying naively to build with all their zest a new nation based on human justice. Later Emmy disappeared completely from the political scene. However she would always remain a pacitist.

In 1930 Weyl was recalled to Goettingen to succeed Hilbert. He too tried various means to obtain a better position for Emmy, but without success. In spite of these fruitless attempts, Emmy had meanwhile become the strongest centre of mathmatical activity in Goettingen. Emmy was one of the rare "slow-developers" in mathematics [5]. In 1920, thirteen years after the submission of her doctoral thesis, the paper collaborated with W. Schmeidler [3] marked the turning-point of her mathematical career and changed the face of algebra. She anticipated the general trends in algebra and strived for the axiomatization of algebra. Among her pupils one could name Krull, Koethe, Deuring, Witt, Fitting, Tsen, Shoda and Levitzki. They were known as "the Noether boys" who formed an active and noisy group in Goettingen. Emmy not only took great interest in their academic progress, but also showed genuine concern over their personal affairs. She maintained an extraordinary, simple and sincere relation with her students.

In 1932 Emmy was invited to address the International Mathematical congress in Zurich. This is one of the highest honour a mathematician could receive in his or her academic life. The significance of her mathematical ideas had at last gained recognition. In the same year, she and Emil Artin were jointly awarded the Ackermann-Teubner Memorial Prize for their contributions to arithmetic and algebra.

Emmy remained in Goettingen until 1933. She was not spared the consequences of the implementation of the Nazi's antisemitic policy, in spite of all the efforts that have been made by her Aryan colleagues and students. Like most other Jewish professors, she was dismissed later that year. Under an arrangement by Weyl she set for America a few months later to take up a position at the woman's college at Bryn Mawr in Pennsylvania.

Bryn Mawr last two years. At Bryn Mawr Emmy adapted herself quickly to the new environment. Soon she was surrounded by her "girls", who held high esteem for her, and looked upon her as a great teacher as well as a sympathetic and sincere companion. Emmy established a very close friendship with Anna Pell Wheeler, chairman of the Bryn Mawr mathematics department, who had spent a few years in Goettingen and had a good understanding of the situation of a woman scholar in Germany. Emmy never failed to introduce Wheeler as her "good friend" to her visitors.

During that period Emmy paid regular visits to the Institute for Advanced study at Princeton to deliver lectures and to conduct seminars. Her visits were usually rounded up by interesting discussions with Hermann Weyl, Oswald Veblen and Richard Brauer, who were then attached to the Institute.

Emmy's stay at Bryn Mawr turned out to be a short one. She died suddenly on 14 April 1935 after an operation at the age of 53. It was not only a loss to her friends but also a great loss to the development of modern algebra.

Teacher and Friend. Emmy was always a faithful and trustworthy friend to her pupils and collaborators. Where mathematics was concerned, she became a strict and non-negotiable judge. This was noted especially in her collaboration of the editing of the "Mathematical Annalen", a job to which she devoted much time. She was humble and most generous in sharing her ideas with others. She would take every opportunity to assist her friends but care little for herself. The position at the University of Torronto offered to Richard Brauer was largely due to her strong recommendation. On the other hand, she remained calm and unconcerned about her own fate during the stormy period of 1933, as remembered vividly by Weyl.

Unlike many other distinguished mathematicians, Emmy was by no means a gifted "class-room" teacher. Her lectures were often ill-organised, technically poor and too erratic. But for the few who had benefited from and were inspired by the lectures, these technical faults could well be over-looked. Her lectures were, so to say, strictly for research students. They attracted a small but faithful audience, consisting of advance students, fellow lecturers and visiting mathematicians. All of them had to exert great concentration in order to follow what was going on the blackboard. Very seldom did she give her theorems in complete form. One chief method of her research was to expound her ideas, which were in a still unfinished state, in her lectures and then to discuss them with her pupils. Every lecture was like a single project. No one could be more delighted than she herself if the project was carried on and completed by her students.

Sometimes Emmy would assist her students in the writing up of their projects by writing the introduction for them. This was because she felt the importance of presenting to the reader (as well as to the author!) the precise purpose and a clear outline of the paper.

Mathematical Contributions and Impact. Emmy was a forerunner in the axiomatization of algebra. The purpose of axiomatization is to provide a means of generalising mathematical results instead of merely solving individual problems under specific conditions. The procedure consists of two parts: the analytical part and the synthetic part. In the analytical part one separates in a natural way the different aspects of a concrete mathematical object by imposing on each of them some set of assumptions. In doing so, the whole breaks into parts each of which is made accessible by virtue of its own relatively narrow and easily surveyable assumptions. The synthetic part is now mechanical, which involves joining the partial results to the complex whole. The art lies in the splitting up the whole into the right partitions. "Emmy carried out this process with masterly skill. She made algebra the El Dorado of axiomatics" [5].

In this connection Emmy's main contribution centred around the general theory of ideals and the investigation of non-commutative algebra. For the former she was very much inspired by Dedekind's theory of modules and ideals, from which she derived from time to time new ideas and methods. Dedekind first introduced the concept of ideals in order to establish the unique factorization theorem in algebraic number fields. By investigating rings with unity element which satisfy the ascending chain condition for ideals Emmy originated the theory of Noetherian rings. The ascending chain condition states that a chain of ideals $A_1 \subset A_2 \subset \cdots A_{i-1} \subset A_i \subset \cdots$ where each A_i contains A_{i-1} properly, necessarily terminates after a finite number of steps. Her abstract theory welded many important concepts of mathematics together, like the classical definition of ideals in algebraic number fields on the one hand, and the polynomial ideals on the other.

After her success in the theory of ideals in commutative rings, Emmy embarked on the study of non-commutative algebras. Here she introduced the notion of a representation space in the representation theory of group rings, by replacing the notion of a matrix by a linear transformation of a vector space. The composition of linear transformations then reflects multiplication in the matrix algebra. This provides us with an excellent geometrical insight into representation theory [1]. A crucial result which thus evolved is the following: Every irreducible R-module of a group ring R corresponds to a maximal ideal of R. With her notion of cross product, she investigated with H. Hasse and R. Brauer the structure of non-commutative algebras. One important result is that every simple algebra over an ordinary algebraic number field is cyclic.

Emmy exerted a considerable influence over a large circle of people. Her contribution could not be assessed by her own papers alone. Many of her ideas and view-points were taken up, further developed and flourished in the works of her students and co-workers. This can be seen in van der Waerden's "Modern Algebra", M. Deuring's "Algebren", and many other research papers with acknowledgements extended to her. Her views on the general theory of ideals also affected the whole essence of the topological research work in Moscow. She was said to be indirectly responsible for the achievements of K.A. Kurosh in algebra [2]. Apart from algebra, frequent applications of Noether's Theorem (though sometimes not explicitly stated) can be found in many physics textbooks, not to mention her contribution to the general theory of relativity.

Emmy felt strongly that to every great mathematical system, even the most abstract, there is a connection with the real world. For her, mathematics is knowledge of the world and not just a game of symbols. "She avidly protested when representatives from those areas of mathematics which are immediately concerned with applications wanted to secure privileges for their practical knowledge" [2]. She was convinced that the substance of human knowledge, including mathematical knowledge, is inexhaustible, at least for many years to come. Nevertheless the new generation would have to work very hard in order to harvest and further develop what has been done so far.

Notes and References

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3. E. Noether and W. Schmeidler, *"Moduln in nicht kommutativen Bereichen, insbesendere aus Differential — und Differenzenausdruecken"*. M. Zeitschrift 8 (1920), 1-35.
4. C. Reid, *Hilbert*, Springer Verlag.
5. H. Weyl, *Emmy Noether*, in Hermann Weyl's *Gesammelte Abhandlungen*, Band III, 425-444, Springer Verlag.
6. B.L. Van der Waerden, *Nachruf auf Emmy Noether*, Math. Annalen 111 (1935), 469-476.

7. The so-called Gordan's Problem deals wtih invariant forms. The existence of a finite basis for binary forms was proved by Gordan in a purely computational way. For a long time no one could extend the result beyond binary forms. Hilbert astonished the mathematical world by showing that such a finite basis must exist for all invariant forms, without having to compute it explicitly.