SOLUTIONS OF PROBLEMS

As the next issue of this Medley will be devoted solely to the proceedings of the Mathematical Symposium held in conjunction with the First Franco-Southeast Asian Mathematical Conference in May 1979 at the University of Singapore — Nanyang University Joint Campus, there will be no Problems and Solutions section in the next issue. In view of this, no problems are proposed in the present issue.

However, members are invited to send interesting problems at secondary school level to Dr. K.N. Cheng, Department of Mathematics, University of Singapore, Singapore 1025.

Solutions to P. 9 — P. 11/78

P9/78. In a certain chess championship between two players A and B, the title goes to the one who first scores six wins; draws do not count. The probability that A wins a game is ¼ and the probability that he draws is ½. If A now leads by five wins to B's four wins, what is the probability that A wins the title eventually? Assume that the games have independent outcomes.

(Tay Yong Chiang)

Solution by Chan Sing Chun.

A needs just one more win to get the title. On the other hand he can afford to lose only once in the subsequent games, otherwise B would have got the title.

The probability that A loses a game is $1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$.

The probability that A wins the title at the k^{th} game after his 5 wins and B's 4 wins, where $k > 1$, is

$$(\tfrac{1}{2})^{k-1}(\tfrac{1}{4}) + \binom{k-1}{k-2}(\tfrac{1}{2})^{k-2}(\tfrac{1}{4})(\tfrac{1}{4})$$

Hence the probability that A wins the title eventually is

$$\tfrac{1}{4} + \sum_{k=2}^{\infty}\left[(\tfrac{1}{2})^{k-1}(\tfrac{1}{4}) + \binom{k-1}{k-2}(\tfrac{1}{2})^{k-2}(\tfrac{1}{4})(\tfrac{1}{4})\right]$$

$$= (\tfrac{1}{4})\sum_{k=0}^{\infty}(\tfrac{1}{2})^{k} + (\tfrac{1}{4})^2 + \sum_{k=0}^{\infty}\binom{k+1}{k}(\tfrac{1}{2})^{k}$$

$$= \tfrac{1}{4}\,\frac{1}{1-\frac{1}{2}} + (\tfrac{1}{4})^2(1-\tfrac{1}{2})^{-2} = \frac{3}{4}$$

(Also solved by proposer)

P10/78. The following "proof" that the alternating group A_5 of degree five is simple appears in *The Fascination of Groups* by F.J. Budden: A_5 has one identity, 24 5-cycles, 20 3-cycles, and 15 double transpositions, making a total of 60 elements. Now we know that if a normal subgroup contains a particular element, then it contains every one of its conjugates. It follows that the order of a normal subgroup of A_5 must be of the form

$$1 + 24n_1 + 20n_2 + 15n_3, \text{ where } n_1, n_2, n_3 \in \{0, 1\}.$$

Also this number must be a factor of 60, the order A_5. This is only possible if $n_1 = n_2 = n_3 = 0$. Hence A_5 has only trivial normal subgroups, i.e. A_5 is simple. Find and correct the error in this "proof".

(Via K. M. Chan)

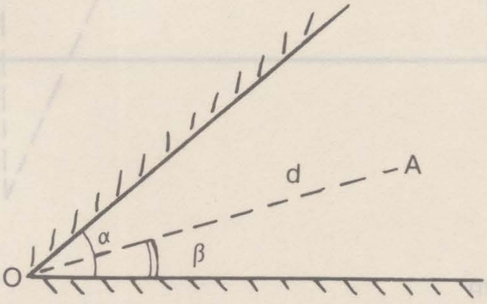
Solution.

The given order formula for a normal subgroup of A_5 is incorrect. As it stands, it implies, for example, that all the 5-cycles of A_5 are conjugates in A_5, which is not the case! This can be seen by computing, say, the centralizer C of (1 2 3 4 5) in A_5. The subgroup C = <(1 2 3 4 5)> is of order five. It follows that (1 2 3 4 5) has exactly $\frac{60}{5}$ = 12 conjugates in A_5, and not 24.

One can verify directly that all the transpositions are conjugates in A_5 and so are all the 3-cycles. Could you perhaps modify the order formula now and give a correct proof that A_5 is simple?

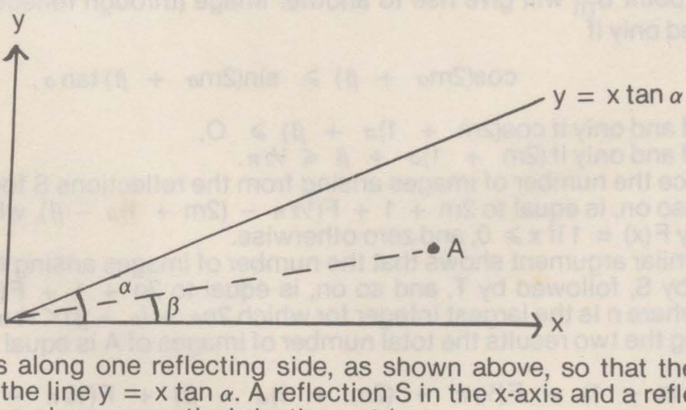
Note: While the elements (1 2 3 4 5) and (1 2 3 5 4) are not conjugates in A_5, they are conjugates in S_5, the symmetric group of degree five. One has (4 5) (1 2 3 4 5) (4 5) = (1 2 3 5 4), where (4 5) lies in S_5 but not in A_5.

*P11/78. Find the total number of images formed by placing a point object A between the reflecting sides of two plane mirrors which intersect at an acute angle α. Assume that A is at a perpendicular distance d from the line of intersection L of the two mirrors, with OA making an angle β with one of the mirrors, O being the foot of perpendicular from A to L (see diagram below).



(Y. K. Leong)

Solution by Proposer.



Take the x-axis along one reflecting side, as shown above, so that the other reflecting side lies on the line y = x tan α. A reflection S in the x-axis and a reflection T in the line y = x tan α are given respectively by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

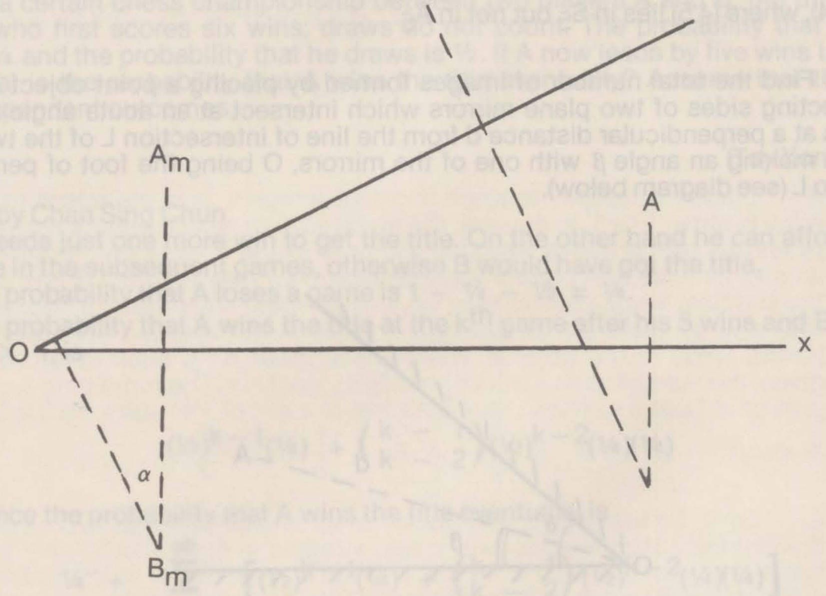
Then the composite transformation TS is a rotation about O through an angle 2α:

$$TS = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

Consider the effect of successive applications on $A = (d\cos\beta, d\sin\beta)$:

$$(TS)^k(A) = \begin{pmatrix} \cos 2k\alpha & -\sin 2k\alpha \\ \sin 2k\alpha & \cos 2k\alpha \end{pmatrix}\begin{pmatrix} d\cos\beta \\ d\sin\beta \end{pmatrix}$$

$$= \begin{pmatrix} d\cos(2k\alpha + \beta) \\ d\sin(2k\alpha + \beta) \end{pmatrix}$$



Let m be the largest integer for which $2m\alpha + \beta < \frac{1}{2}\pi$, and let

$A_m = (TS)^m(A)$, $B_m = S(TS)^m(A)$.

Then B_m is the point $(\cos(2m\alpha + \beta), -c\sin(2m\alpha + \beta))$,

The point B_m will give rise to another image (through reflection in the line $y = x \tan\alpha$) if and only if

$$\cos(2m\alpha + \beta) \geqslant \sin(2m\alpha + \beta)\tan\alpha,$$

i.e. if and only if $\cos(2m + 1)\alpha + \beta) \geqslant O$,

i.e. if and only if $(2m + 1)\alpha + \beta \leqslant \frac{1}{2}\pi$.

Hence the number of images arising from the reflections S followed by T, followed by S, and so on, is equal to $2m + 1 + F(\frac{1}{2}\pi - (2m + 1)\alpha - \beta)$, where the function F is defined by $F(x) = 1$ if $x \geqslant 0$, and zero otherwise.

A similar argument shows that the number of images arising from the reflections T followed by S, followed by T, and so on, is equal to $2n + 1 + F(\frac{1}{2}\pi - (2n + 1)\alpha - (\alpha - \beta))$, where n is the largest integer for which $2n\alpha + (\alpha - \beta) < \frac{1}{2}\pi$.

Combining the two results the total number of images of A is equal to

$$2(m + n + 1) + F(\tfrac{1}{2}\pi - (2m + 1)\alpha - \beta) + F(\tfrac{1}{2}\pi - (2n + 2)\alpha + \beta)$$

where m, n are the largest integers less than

$$(\pi/4\alpha) - \beta/(2\alpha), (\pi/4\alpha) - (\alpha - \beta)/(2\alpha) \text{ respectively.}$$