THE CHINESE METHOD OF SOLVING POLYNOMIAL EQUATIONS OF SEVERAL VARIABLES

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Would you like to solve the following set (A) of four equations in the four unknowns x, y, z and u by reducing them to an equation of one variable in u?

$$(A)\quad \begin{cases} -2y + x + z = 0 \\ -xy^2 + 4y - x^2 + 2x + xz + 4z = 0 \\ y^2 + x^2 - z^2 = 0 \\ 2y + 2x - u = 0 \end{cases}$$

In 1303, Zhu Shijie 朱世杰 showed in his book *Siyuan yujian* 四元玉鉴 (Jade mirror of the four unknowns) [1], [2], [3] a method of eliminating the three unknowns x, y and z and reducing the equations to the following single equation in u,

$$4u^2 - 7u - 686 = 0.$$

In so doing, he established the lead by the Chinese for over 450 years in the elimination theory of polynomial equations of several variables.

In Europe, it was Etienne Bezout [4] who initiated the study of solving a pair of polynomial equations in two unknowns, $f(x, y) = 0$ and $g(x, y) = 0$, of degree higher than one. In 1764, he presented a paper, where he showed his method of eliminating one unknown from the two equations by multiplying $f(x, y)$ and $g(x, y)$ by suitable polynomials $F(x)$ and $G(x)$ respectively, in order to obtain an equation in one unknown of the form

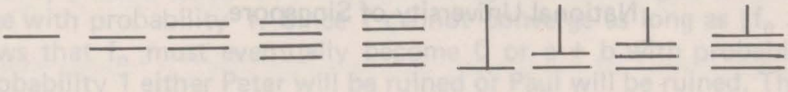
$$R(y) = F(x)f(x,y) + G(x)g(x,y) = 0.$$

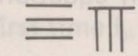
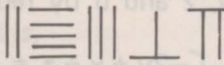
When the above set (A) of four equations was reduced to an equation in one unknown, the problem could then be solved as the mathematicians in thirteenth century China were familiar with a method of finding a positive root of a polynomial equation of any degree in one variable [5]. In contrast to this, it was not till the first half of the sixteenth century that mathematicians in the West such as Scipione dal Ferro, Niccolo Fontana (also known as Tartaglia) and Jerome Cardan made a breakthrough in the solution of cubic equations [6]. The method used by the Chinese to find a positive solution to a numerical equation of higher degree has been confirmed by historians of mathematics to be similar to the method used by W. G. Horner in 1819 [7], [8].

In the opening pages of the *Siyuan yujian*, Zhu Shijie gave four problems showing how equations were set up, and in the case of equations of more than one variable, he displayed brief methods of how they could be reduced to an equation in one unknown. Computations were done with counting rods on a counting board.

The digits one to nine were represented by counting rods either in the from

or

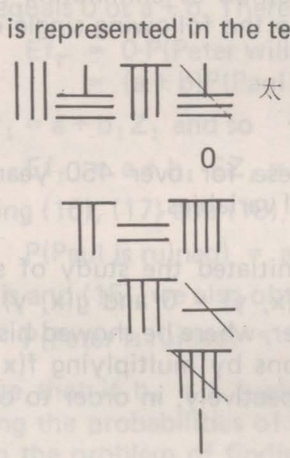


The form of representation depended on the place-value of each digit. For instance, 48 was represented by  and 25367 by . A rod placed diagonally across the last non-zero digit would indicate that the number was negative.

In the first example of Zhu's book, the derived polynomial equation is of the form

$$x^5 - 9x^4 - 81x^3 + 72x^2 - 3888 = 0$$

and this is represented in the text as follows:



which means

-3888 *tai*
0
729
-81
-9
1

The character *tai* 太 indicates the row where the constant term is placed. The coefficients of the increasing positive powers of x are placed in the successive rows below, while the coefficients of the negative powers are put in the rows above. The counting board notation does not distinguish between a polynomial f(x) and its corresponding equation f(x) = 0. Whichever is meant, has to be ascertained from the context.

For an expression in two unknowns, their coefficients are displayed on the counting board in a two-dimensional array as follows:

			$1/x^2$		
	y^2/x	y/x	$1/x$		
y^3	y^2	y	*tai*	$1/y$	$1/y^2$
	xy^2	xy	x	x/y	
	x^2y^2	x^2y	x^2	x^2/y	
		x^3y	x^3		

Zhu displayed five such representations to explain his second problem. They are

−2	0	*tai*		2	0	*tai*
−1	2	0		−1	2	0
0	2	0		0	0	0
0	0	1		0	0	1
	(i)				(ii)	

tai	*tai*	
8	0	−8
4	2	−2
	1	1
(iii)	(iv)	(v)

In this problem, it is required to solve a set of two equations in the form

$$(-x-2)y^2 + (2x^2 + 2x)y + x^3 = 0$$

represented by (i) and

$$(-x+2)y^2 + 2xy + x^3 = 0$$

represented by (ii).

These two equations are reduced to an equation in one unknown $x^2 - 2x - 8 = 0$ represented by (v), through the subtraction of (iii) from (iv) which in algebraic notations are

$$4x^2 + 8x \quad \text{(iii)}$$

and
$$x^3 + 2x^2 \quad \text{(iv)}.$$

The procedure given by Zhu is so brief that there are a number of interpretations to derive (iii) and (iv). One of the processes is given below. The algebraic equivalence is written on the right.

Step 1. Subtract the corresponding terms of (i) from those of (ii) to obtain

4	0	*tai*	
0	0	0	$4y^2 - 2x^2y = 0$
0	−2	0	
	(vi)		

Step 2. Reduce (vi) by one column and divide by 2.

2	*tai*	
0	0	$2y - x^2 = 0$
0	−1	
	(vii)	

Step 3. Increase (vii) by one row.

0	*tai*
2	0
0	0
0	−1

(viii)

$$2xy - x^3 = 0$$

Step 4. Add the corresponding terms of (i) and (viii) to give

−2	0	*tai*
−1	4	0
0	2	0

(ix)

$$(-x-2)y^2 + (2x^2 + 4x)y = 0$$

Step 5. Reduce (ix) by one column to give

−2	*tai*
−1	4
0	2

(x)

$$(-x-2)y + 2x^2 + 4x = 0$$

Step 6. Cross-multiply the first column of (vii) with the second column of (x) to give (iii), and the second column of (vii) with the first column of (x) to give (iv).

2		*tai*		*tai*
0	x	4	=	8
0		2		4

(iii)

$$2(2x^2 + 4x) = 4x^2 + 8x$$

−2		*tai*		*tai*
−1	x	0	=	0
0		−1		2
				1

(iv)

$$(-x-2)(-x^2) = x^3 + 2x^2$$

Step 7. Subtract the corresponding terms of (iii) from those of (iv) and reduce the result by one row to obtain (v).

Analysing the above method which is computed on a counting board, we can write down the procedure in general terms as follows:

In the given arrays of numbers, the first column from the right represents a polynomial $f_0(x)$, the second and third columns represent polynomials $f_1(x)y$ and

$f_2(x)y^2$ respectively, and so on. Thus the above arrays (i) and (ii) represent equations of the form

$$f_2(x)y^2 + f_1(x)y + f_0(x) = 0$$

and $$g_2(x)y^2 + g_1(x)y + g_0(x) = 0.$$

From these two equations, the terms in y^2 are eliminated to obtain

$$p_1(x)y + p_0(x) = 0, \quad \text{see (vii).}$$

Another equation of similar form is obtained when this equation is applied to one of the above two equations. We have

$$q_1(x)y + q_0(x) = 0, \quad \text{see (x).}$$

In order to eliminate y from the last two equations, there is a cross-multiplication of the columns in arrays (vii) and (x), or in other words,

$$p_1(x)q_0(x) - p_0(x)q_1(x) = 0$$

This equation is represented by (v).

We have so far explained polynomials in two unknowns. As the number of unknowns in the polynomials increases so does the intricacy in the process of elimination. The representation of the coefficients of three unknowns on a counting board is shown below. Note that now the coefficients of terms like xyz have to be superimposed.

y^3	y^2	y	*tai*	z	z^2	z^3
	xy^2	xy	x	xz	xz^2	
		xyz				
	x^2y^2	x^2y	x^2	x^2z	x^2z^2	
			x^3			

For a polynomial expression involving four unknowns, the whole space of the counting board is occupied as shown below.

u^3y^3	u^3y^2	u^3y	u^3	u^3z	u^3z^2	u^3z^3
u^2y^3	u^2y^2	u^2y	u^2	u^2z	u^2z^2	u^2z^3
uy^3	uy^2	uy	u	uz	uz^2	uz^3
				yz		
y^3	y^2	y	*tai*	z	z^2	z^3
		xu				
xy^3	xy^2	xy	x	xz	xz^2	xz^3
x^2y^3	x^2y^2	x^2y	x^2	x^2z	x^2z^2	x^2z^3
x^3y^3	x^3y^2	x^3y	x^3	x^3z	x^3z^2	x^3z^3

As an illustration of how this representation is used, we give below a translation of Zhu's fourth problem. This problem involves four equations in four unknowns which in algebraic notations are equivalent to the set (A) of four equations in the first paragraph of this article.

Zhu's fourth problem states

"The product of the five differences and altitude [of a right-angled triangle] equals the sum of the square of the hypotenuse and the product of the base and hypotenuse. It is also given that the quotient of the five sums and the base equals the square of the altitude minus the difference of the hypotenuse and base. Find the *huangfang* 黄方 plus the sum of the base, altitude and hypotenuse.

Answer: 14 *bu* 步 ".

If we take x = base, y = altitude and z = hypotenuse, then the five differences are $y - x$, $z - x$, $z - y$, $z - (y - x)$ and $(x + y) - z$. The five sums are $x + y$, $y + z$, $z + x$, $z + (x - y)$ and $(x + y) + z$. The *huangfang* is a technical term for $x + y - z$. The two equations of the problem are

$$\{y - x + z - x + z - y + z - (y - x) + x + y - z\}\, y = z^2 + xz$$

and $$(x + y + y + z + z + x + z + x - y + x + y + z)/x = y^2 - (z - x),$$

and it is required to find $x + y - z + x + y + z$ which we represent by u. These equations are solved together with the equation derived from the right-angled triangle, namely

$$x^2 + y^2 = z^2.$$

In showing the working of the problem, Zhu gave the following arrays to show their representations of polynomials on a counting board. The algebraic interpretations [9] are written besides them.

−2	*tai*	1			First equation
0	1	0			$-2y + x + z = 0.$
0	4	*tai*	4		Second equation
−1	0	2	1		$-xy^2 + 4y - x^2 + 2x + (x + 4)z = 0$
0	0	−1	0		
1	0	*tai*	0	−1	From the right-angled triangle
0	0	0	0	0	$y^2 + x^2 - z^2 = 0.$
0	0	1	0	0	
0	−1	0			To find
2	*tai*	0			$u = 2y + 2x.$
0	2	0			

2 −8 28 *tai* 0 −1 6 −2 0 0 0 −1	An elimination process reduces the above equations to the next two equations after x and u are interchanged $2y^3 + (-x-8)y^2 + (6x + 28)y - x^2 - 2x = 0$
−7 *tai* 0 2	$-7y + 2x = 0.$
0 294 8 3 0 −4	The next step is to reduce the arrays to two-column ones. $8xy - 4x^2 + 3x + 294 = 0.$
0 0 16	A process of cross-multiplication produces the inner column $16x^2$
−2058 −21 28	and the outer column $28x^2 - 21x - 2058$
−686 −7 4	from which the quadratic equation $4x^2 - 7x - 686 = 0$ is obtained.

Zhu did not set problems involving polynomials of more than four variables for the obvious reason that the space on a counting board was limited. To solve such problems he would have to construct a three-dimensional board! Besides the four problems at the beginning of the book, the *Siyuan yujian* has 288 problems, of which 36 are connected with two equations in two unknowns, 13 are involved with three equations in three unknowns and 7 with four equations in four unknowns.

References

[1] Zhu Shijie. *Siyuan yujian* (Jade mirror of the four unknowns). China, 1303.

[2] Hoe, John. *Le Systemes D'Equations Polynômes Dans Le Siyuan Yujian (1303).* Paris: College de France, 1977.

[3] Hoe, John. "The jade mirror of the four unknowns — some reflections". *Math. Chronicle,* 1978, *7,* 125–156.

[4] Kline, Morris. *Mathematical Thought from Ancient to Modern Times.* New York: Oxford University Press, 1972, p. 608.

[5] Lam Lay Yong. "The Chinese connection between Pascal triangle and the solution of numerical equations of any degree." *Historia Mathematica,* 1980, *7,* 407–424.

[6] Kline, Morris. *Op. cit.* p. 263.

[7] Mikami, Yoshio. *The Development of Mathematics in China and Japan.* Chelsea Publishing Co, 1974 (2nd edition), p. 77.

[8] Needham, Joseph. *Science and Civilisation in China.* Vol. 3. Cambridge: Cambridge University Press, 1959, pp. 43, 126.

[9] Lam Lay Yong. "Chinese polynomial equations in the thirteenth century." Shanghai: *Zhonghua wen shi lungcong* (to appear).