THE GOLDBACH CONJECTURE*

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The Goldbach conjecture was proposed in a letter from Goldbach to Euler in 1742. It may be stated as follows.

(1) $2n = p + p'$,
(2) $2n + 1 = p + p' + p''$.

Hereafter we always use p, p′, p″, . . . to denote prime numbers.

Two centuries have passed, but the conjectures are still not proved or disproved. It was shown by computer that the conjecture is true for $n \leqslant 3 \times 10^7$.

In the 20th century, many great mathematicians were attracted by this conjecture. In 1900, D. Hilbert gave a famous speech in an international mathematical conference, in which he proposed 23 problems to mathematicians. The Goldbach conjecture is a part of his 8th problem and the other part is the Riemann hypothesis. G.H. Hardy said that the Goldbach conjecture is one of the most difficult problems in mathematics. However in the past 70 years, many remarkable achievements were obtained concerning the Goldbach conjecture.

(1) Circle Method

The circle method was proposed by Hardy, Littlewood and Ramanujan in 1915. Since

$$\int_0^1 e^{2\pi i \alpha x} dx = \begin{cases} 1, \text{ if } \alpha = 0, \\ 0, \text{ if } \alpha \neq 0, \end{cases}$$

the number of solutions $\mathcal{R}(n)$ of (1) is clearly equal to

$$\mathcal{R}(n) = \int_0^1 \Big(\sum_{p \leqslant 2n} e^{2\pi i \alpha p} \Big)^2 e^{-2\pi i (2n) \alpha} d\alpha.$$

The arc of the integral may be divided into two parts; the major arc $\mathfrak{M}$ and the minor arc m. Roughly speaking, the major arc contains those subintervals which contain the rationals h/q with comparatively small denominators, and the remaining part of [0, 1] is called the minor arc.

It may be proved that

$$\int_{\mathfrak{M}} \Big(\sum_{p \leqslant 2n} e^{2\pi i \alpha p} \Big)^2 e^{-2\pi i (2n)\alpha} d\alpha \sim c \prod_{\substack{p | 2n \\ p > 2}} \frac{p-1}{p-2} \cdot \frac{n}{\ln^2 n}, \quad c > 0$$

but we cannot prove that

$$\int_m \Big(\sum_{p \leqslant 2n} e^{2\pi i \alpha p} \Big)^2 e^{-2\pi i (2n)\alpha} d\alpha = 0\Big(\frac{n}{\ln^2 n}\Big).$$

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It is easily seen that (2) is a cosequence of (1). In fact, if (1) is true, then

$$2n + 1 - 3 = p + p',$$
$$2n + 1 = 3 + p + p',$$

and the assertion follows.

Let R(n) denote the number of solutions of (2). Then

$$R(n) = \int_0^1 \Big(\sum_{p \leqslant 2n+1} e^{2\pi i \alpha p} \Big)^3 e^{-2\pi i (2n+1)\alpha} d\alpha.$$

We have

$$\int_{\mathfrak{M}} \Big(\sum_{p \leqslant 2n+1} e^{2\pi i \alpha p} \Big)^3 e^{-2\pi i (2n+1)\alpha} d\alpha \sim C \frac{n^2}{\ln^3 n}, \quad C > 0.$$

Under the assumption of the Grand Riemann hypothesis (GRH), Hardy and Littlewood proved that the contribution on the minor arc is $o(\frac{n^2}{\ln^3 n})$. It should be noticed that the original method of Hardy and Littlewood is slightly different from the method used here and this is the modified method due to I.M. Vinogradov. Hence under the assumption of (GRH), Hardy and Littlewood proved the conjecture (2) for sufficiently large n.

The improved hypothesis in Hardy and Little wood's proof of (2) was removed by Vinogradov in 1937. Hence he obtained

Theorem 1 (Vinogradov). Every sufficiently large odd integer is a sum of three primes.

Borozdki proved that (2) is true for

$$n > n_0 = e^{e^{e^{17}}}$$

By the method of Hardy-Littlewood-Vinogradov, Davenport, Heilbronn, Hua and Van de Corput proved independently the following

Theorem 2 Let M(x) denote the number of even integers 2n such that $2n \leqslant x$ and cannot be represented as a sum of two primes. Then

$$M(x) = o(x).$$

(2) Sieve Method

The historical origin of the sieve method is the well-known "Sieve of Eratosthenes". Eratosthenes noted that the prime number between $n^{1/2}$ and n can be isolated by removing from the sequence 2, 3, . . . , n every number which is multiple of a prime not exceeding $n^{1/2}$.

Let ℓ be an integer and $\prod_{p \leqslant (2n)^{\frac{1}{\ell+1}}} p = \prod$.

Let $$P_\ell(n) = \sum_{\substack{(m(2n-m), \prod) = 1 \\ m \leqslant 2n}} 1.$$

Provided that we can obtain a positive lower estimation for $P_\ell(n)$ when n is large, it will follow that every large even integer 2n is a sum of two numbers each being a product of at most ℓ prime factors.

The above proposition is denoted by (ℓ, ℓ). Similarly we may define (ℓ, m) for $\ell \neq m$.

V. Brun was the first to prove (9, 9).

Theorem 3 (Brun). (9, 9).

Brun's method and his result was improved by several mathematicians; for example

(7, 7)	(H. Rademacher, 1924),
(6, 6)	(T. Estermann, 1932),
(5, 7), (4, 9), (3, 15), (2, 366)	(G. Ricci, 1937),
(5, 5), (4, 4)	(A. Buchstab 1938 – 1940),
(a, b), $(a + b \leqslant 6)$	(P. Kuhn, 1953 – 1954).

By a combination of the new sieve method of A. Selberg with the method of Brun, Buchstab and Kuhn, Wang Yuan proved

(3, 4), (3, 3), (a, b) $(a + b \leqslant 5)$, (2, 3) (1956 – 1957),

in which (3, 3) was obtained by A.I. Vinogradov independently.

If we consider the sum

$$Q_\ell(n) = \sum_{\substack{(2n-p,\ \prod) = 1 \\ p \leqslant 2n}} 1$$

then it follows by a positive lower estimation of $Q_\ell(n)$ that every large even integer is the sum of a prime and a product of at most ℓ primes, or simply $(1, \ell)$.

In 1932, Estermann first proved (1, 6) under the assumption of (GRH). Without any improved hypothesis, A. Renyi proved in 1948 by means of the large sieve of U.V. Linnik and Renyi:

Theorem 4 (Renyi). (1, c), where c is a constant.

Estermann's result was improved by Wang Yuan to (1, 3) under the (GRH), and Wang Yuan also pointed out that (GRH) in the proof of (1, 3) can be replaced by a mean value theorem concerning the primes in arithmetic progressions. However, Barban and Pan Ching-Tong proved independently a weaker mean value theorem in 1962, namely, for any given $\epsilon > 0$ and $A > 0$,

$$\sum_{k \leqslant x^{\frac{3}{8}-\epsilon}} \max_{(\ell, k)=1} \left| \pi(x,k,\ell) - \frac{1}{\phi(k)} \int_\epsilon^x \frac{dt}{\ln t} \right| = 0\left(\frac{x}{\ln^A x}\right),$$

where $\pi(x, k, \ell)$ denotes the number of primes $\leqslant x$ and $\equiv \ell \pmod k$ and $\phi(k)$ denotes the Euler function. Therefore they proved (1, 4).

In 1965, E. Bombieri proved a remarkable mean value theorem

$$\sum_{k \leqslant x^{1/2}/\ln^A x} \max_{(\ell, k) = 1} \left| \pi(x, k, \ell) - \frac{1}{\phi(k)} \int_\epsilon^x \frac{dt}{\ln t} \right| = O\left(\frac{x}{\ln^B x}\right).$$

where B is a given constant and A = A(B). Bombieri's theorem may be used to replace the improved hypothesis in the proof of (1, 3).

Finally, Chen Jing Run introduced some new idea to treat the Goldbach conjecture by the sieve method and he succeeded in 1966 to prove the following

Theorem 5 (Chen Jing-Ren), (1, 2).

(GRH) has several equivalent forms, one of which may be stated as follows.

For any given k, ℓ such that $(k, \ell) = 1$, the relation

$$\pi(x, k, \ell) = \frac{1}{\phi(k)} \int_\epsilon^x \frac{dt}{\ln t} + O(\sqrt{x} \ln x)$$

holds.

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