INTERSCHOOL MATHEMATICAL COMPETITION 1982

PART A

Saturday, 3 July 1982 0900 – 1000

Attempt as many questions as you can. Circle your answers on the Answer Sheet provided.

Each question carries 5 marks.

1. For $n \geqslant 100$,

$$\sqrt{(1 + 2\sqrt{(1 + 3\sqrt{(1 + \ldots + (n-1)\sqrt{(1 + n\sqrt{(n+2)^2})}\ldots)}}}$$

is equal to

(a) 2,
(b) 3,
(c) $3 + n^{1/n}$,
(d) $3 + (n+2)^{1/n}$,
(e) none of the above.

2. The number 104060401 is a product of

(a) 4 primes,
(b) 5 primes,
(c) 6 primes,
(d) 7 primes,
(e) 8 primes.

3. Let $A = \{ n \mid n \text{ is a positive integer and } 2^n \leqslant n^2 \}$ Then

(a) A is empty,
(b) A has exactly 2 elements,
(c) A has exactly 3 elements,
(d) A is the set of all positive integers greater than 1,
(e) none of the above.

4. The value of $\int_0^{\pi/2} \sin^5 x/(\sin^5 x + \cos^5 x)dx$ is

(a) π,
(b) $\pi/2$,
(c) $\pi/3$,
(d) $\pi/4$,
(e) $\pi/5$.

5. Given that $1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \to \gamma$ as $n \to \infty$, where γ is a constant, then

$$\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \to c, \text{ where}$$

(a) $c = 0$,
(b) $c = 2 \log 2$,
(c) $c = 1/\log 2$,
(d) $c = \log 2$,
(e) the value of c cannot be determined from the given data.

6. Let $A = (1-x)(1-x^3)(1-x^5) \ldots (1-x^{2n-1}) \ldots,$

$B = (1+x)(1+x^3)(1+x^5) \ldots (1+x^{2n-1}) \ldots,$

$C = (1+x^2)(1+x^4)(1+x^6) \ldots (1+x^{2n}) \ldots,$

where $|x| < 1$. Then ABC is equal to

(a) $\frac{1}{3}$,
(b) $\frac{1}{2}$,
(c) $1/\sqrt{2}$,
(d) 1,
(e) none of the above.

7. Let f be any function defined on the set of real numbers such that $(f(x))^2 = 4x^2$ for all x. Then

(a) for every real number a, we have

$$\lim_{h \to 0} f(a+h) = f(a),$$

(b) there does not exist a real number a such that

$$\lim_{h \to 0} f(a+h) = f(a),$$

(c) there exists exactly one real number a such that

$$\lim_{h \to 0} f(a+h) = f(a),$$

(d) there exists at least one real number a such that

$$\lim_{h \to 0} f(a+h) = f(a),$$

(e) none of the above is true.

8. The values of x (real) for which the inequality

$$\left(\frac{1}{2}\right)^{\sqrt{(x^6 - 2x^3 + 1)}} < \left(\frac{1}{2}\right)^{1 - x}$$

holds are

(a) $-1 < x < 1$,
(b) $x < -1,\ 0 < x < 1,\ x > 1$,
(c) $x < -1,\ -1 < x < 0,\ x > 1$,
(d) $x < -1,\ -1 < x < 1,\ x > 1$,
(e) none of the above.

9. Let a, b, c, d be numbers from the set $\{-1, 0, 1\}$. Let m be the number of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let n be the number of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $m - n$ is equal to

(a) 0,
(b) 3,
(c) 6,
(d) 9,
(e) 27.

10. An insect moves along the sides of a square ABCD (see Fig. 1). From a given vertex of the square, the insect moves to each of the two adjacent vertices with probability $\frac{1}{2}$. Starting from vertex A, the insect will return to A in exactly 20 moves with probability

(a) $\left(\frac{1}{4}\right)^{20}$,

(b) $\left(\frac{1}{4}\right)^{10}$,

(c) $(\frac{1}{2})^8$,

(d) $(\frac{1}{2})^{10}$,

(e) none of the above.

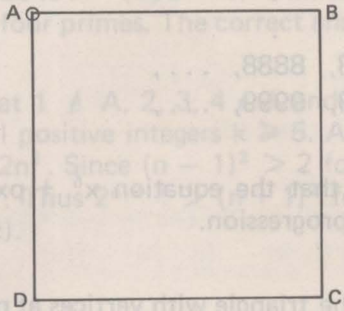


Fig. 1

PART B

Saturday, 3 July 1982 — 1000 – 1200

Attempt as many questions as you can.

Each question carries 25 marks.

1. If p is an odd prime and

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{p-1} = \frac{a}{b},$$

where a, b are positive integers with no common factor greater than 1, show that p divides a.

2. For any real number a, write [a] for the greatest integer less than or equal to a.

Suppose x and y are positive irrational numbers such that $\frac{1}{x} + \frac{1}{y} = 1$. Show that the collection of integers

$$[x], [y], [2x], [2y], \dots, [nx], [ny], \dots$$

includes every positive integer once and once only.

3. Show that none of the numbers in the following sequences is a perfect square:

11, 111, 1111, . . . ,
22, 222, 2222, . . . ,
33, 333, 3333, . . . ,
. . .
. . .
. . .
88, 888, 8888, . . . ,
99, 999, 9999, . . . ,

4. Determine p and q so that the equation $x^6 + px^4 + qx^2 - 225 = 0$ has six real roots in arithmetic progression.

5. If the circumcircle of the triangle with vertices at $p = (-\frac{1}{2}c,\ 0)$, $Q = (\frac{1}{2}c,\ 0)$, $R = (x, y)$ has unit radius, show that $\overline{QR}^2 . \overline{RP}^2 = 4y^2$.

Let ABC be a triangle in the xy-plane such that its vertices have integer co-ordinates. If the circumcircle of ABC has unit radius, prove that $abc \geqslant 2$, where a = BC, b = CA, c = AB.

6.

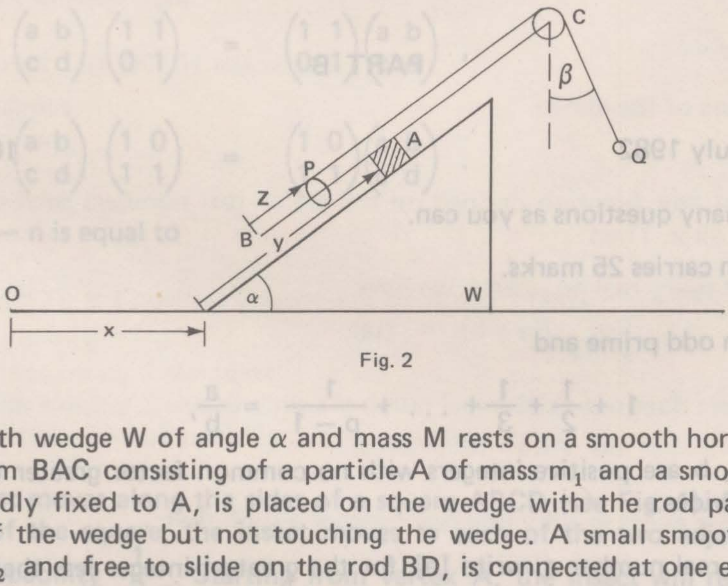


Fig. 2

A smooth wedge W of angle α and mass M rests on a smooth horizontal table. A system BAC consisting of a particle A of mass m_1 and a smooth light rod BC, rigidly fixed to A, is placed on the wedge with the rod parallel to the slope of the wedge but not touching the wedge. A small smooth ring P, of mass m_2 and free to slide on the rod BD, is connected at one end to a light inextensible string. The string passes over a light pulley C and supports a particle Q of mass m_3 at the other end. If in the ensuing motion, the string CQ makes a constant angle $\beta (< \alpha)$ with the vertical, write down five differential equations involving the four variables x, y, z (see Fig. 2) and T, the tension in the string.

[The equations should not contain the reactions.]

SOLUTIONS TO PART A

1. At each step one evaluates $\sqrt{1 + k(k + 2)}$, which is equal to (k + 1). At the last step k = 2. Thus the correct answer is (b).

2. We observe that $104060401 = (100 + 1)^4$. Since 101 is a prime, the given number is a product of four primes. The correct answer is (a).

3. It is easily checked that $1 \notin A$, $2, 3, 4 \in A$ and $5 \notin A$. We use induction to show that $k \notin A$ for all positive integers $k \geqslant 5$. Assume that $2^n > n^2$, $n \geqslant 5$. Then $2^{n+1} = 2.2^n > 2n^2$. Since $(n - 1)^2 > 2$ for $n \geqslant 5$, we get $n^2 > 2n + 1$, so that $2n^2 > (n + 1)^2$. Thus $2^{n+1} > (n + 1)^2$ for all positive integers $n \geqslant 5$. The correct answer is (c).

4. We note that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x + \cos^5 x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^5 x\, dx}{\sin^5 x + \cos^5 x} = \int_0^{\frac{\pi}{2}} 1\, dx = \frac{\pi}{2}.$$

$$\text{But} \int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x + \cos^5 x}\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^5 (\frac{\pi}{2} - x)}{\cos^5 (\frac{\pi}{2} - x) + \sin^5 (\frac{\pi}{2} - x)}\, dx$$

$$= \int_{\frac{\pi}{2}}^{0} \frac{\cos^5 y}{\cos^5 y + \sin^5 y}\, d(-y), \text{ by setting } y = \frac{\pi}{2} - x,$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^5 y}{\sin^5 y + \cos^5 y}\, dy.$$

$$\text{Hence} \int_0^{\frac{\pi}{2}} \frac{\sin^5 y}{\sin^5 y + \cos^5 y}\, dy = \frac{\pi}{4}.$$

The correct answer is (d).

5. By the given condition $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n} - \log 2n$ $\to \gamma$ as $n \to \infty$. i.e. $1 + \frac{1}{2} + \ldots + \frac{1}{2n} - \log 2 - \log n \to \gamma$, or $(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n) + (\frac{1}{n+1} + \ldots + \frac{1}{2n} - \log 2) \to \gamma$. Now, since $1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \to \gamma$ as $n \to \infty$, we get $\frac{1}{n+1} + \ldots + \frac{1}{2n} - \log 2 \to 0$ as $n \to \infty$. The correct answer is therefore (d).

6. One computes directly that ABC = (AB)C

$$= \left[\prod_{n=0}^{\infty} (1 - x^{2(2n+1)})\right] \left[\prod_{n=1}^{\infty} (1 + x^{2n})\right]$$

$$= (1-x^2)(1+x^2)(1+x^4)(1-x^6)(1+x^6)(1+x^8)\ldots\ldots$$

$$= \prod (1-x^{12})(1-x^{16})\ldots\ldots$$

$$\to 1 \text{ as } n \to \infty, \text{ since } |x| < 1.$$

The correct answer is therefore (d).

7. Taking square roots, $f(x) = 2x$ or $f(x) = -2x$. As $\pm 2x \to 0$ as $x \to 0$, and $f(0) = 0$, we have at least at $a = 0$, $\lim_{h \to 0} f(a+h) = f(a)$.

The correct answer is therefore (d).

Examples

(1) Let $f(x) = 2x$ for all real x. Then for every real number a, $\lim_{h \to 0} f(a+h) = f(a)$.

(2) Let $f(x) = \begin{cases} 2x, & x \text{ rational,} \\ -2x, & x \text{ irrational.} \end{cases}$

Then f(x) is continuous at only a = 0.

8. The given inequality is equivalent to $(x^6 - 2x^3 + 1)^{\frac{1}{2}} > 1 - x$ ..... (1). L. H. S. equals $((x^3 - 1)^2)^{\frac{1}{2}} = |x^3 - 1|$. Clearly (1) holds for $x > 1$. When $x \leqslant 1$, then $|x^3 - 1| = 1 - x^3$ and (1) becomes $1 - x^3 > 1 - x$. Solving, we get $x < -1$ or $0 < x < 1$. Thus the correct answer is (b).

9. By direct computation, matrices which commute with $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ have the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Thus m = 9. And matrices which commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are of the form $\begin{pmatrix} a & o \\ o & a \end{pmatrix}$. Thus n = 3. The correct answer is (c).

10. The number of different paths for which the insect will return to A for the first time in 20 moves in 2^{10}. The probability for each path to occur is $(\frac{1}{2})^{20}$, thus the required probability is $2^{10} \cdot (\frac{1}{2})^{20} = (\frac{1}{2})^{10}$. The correct answer is therefore (d).

SOLUTIONS TO PART B

1. Since p is an odd prime, the given sum is equal to

$$\left(1 + \frac{1}{p-1}\right) + \left(\frac{1}{2} + \frac{1}{p-2}\right) + \ldots + \left(\frac{1}{\frac{p-1}{2}} + \frac{1}{\frac{p+1}{2}}\right)$$

$$= \frac{p}{p-1} + \frac{p}{2(p-2)} + \ldots + \frac{p}{\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)}$$

Clearly p does not divide any of the factors occuring in the denominators. It follows that the above sum can be expressed in the form $\frac{a}{b}$, where a and b have no common factor greater than 1 and p divides a.

2. Let $A = \{ [x], [2x], \ldots, [nx], \ldots \}$
and $B = \{ [y], [2y], \ldots, [ny], \ldots \}$.

Since x and y are both positive and $\frac{1}{x} + \frac{1}{y} = 1$, we have $x > 1$ and $y > 1$. Thus elements in A (or B) are distinct.

We claim that $A \cap B = \emptyset$. Suppose there exists a positive integer k in $A \cap B$. Then we can find positive integers p and q such that $k < px < k + 1$, $k < qy < k + 1$ (since x and y are positive irrationals).

Hence $\frac{p}{k+1} < \frac{1}{x} < \frac{p}{k}$, $\frac{q}{k+1} < \frac{1}{y} < \frac{q}{k}$.

Adding the above inequalities, we have

(1) $$\frac{p+q}{k+1} < 1 < \frac{p+q}{k}.$$

(1) implies $k < p + q < k + 1$ which is absurd because p, q and k are positive integers. Hence $A \cap B = \emptyset$.

We next show that every positive integer is in $A \cup B$. Suppose there exists a positive integer m not in $A \cup B$. Then we can find positive integers r and s such that

$$rx < m, \ (r+1)x > m+1;$$

$$sy < m, \ (s+1)y > m+1.$$

Hence $\frac{r+1}{m+1} < \frac{1}{x} < \frac{r}{m}$, $\frac{s+1}{m+1} < \frac{1}{y} < \frac{s}{m}$.

Adding the above inequalities, we get

$$(2) \qquad \frac{r+s+2}{m+1} < 1 < \frac{r+s}{m} .$$

Again (2) implies $r + s + 1 < r + s$ which is impossible. Hence every positive integer occurs once and only once in $A \cup B$.

3. Since the last digit of a perfect square is 0, 1, 4, 5, 6 or 9, numbers in { aa, aaa, . . . } are not perfect squares when a = 2, 3, 7, 8.

   Since the square of an even integer is of the form 4m and that of an odd integer is of the form 4n + 1 where m, n are positive integers, numbers in { aa, aaa, . . . } are not perfect squares when a = 1, 5, 6.

   Lastly, for a = 4 or 9, we have aa . . . a = a. (11 . . . 1) Since a is a perfect square and 11 . . . 1 is not, aa . . . a is not a perfect square.

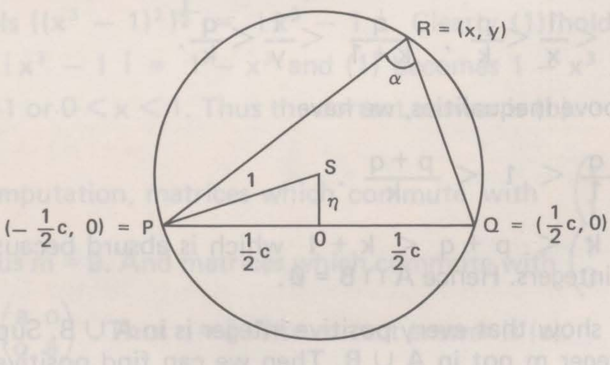
4. We observe that α is a root of the given equation if and only if $-\alpha$ is a root and that O is not a root. Let the smallest positive root be α. Then the six roots are $-5\alpha, -3\alpha, -\alpha, \alpha, 3\alpha, 5\alpha$.

   Hence $(-5\alpha).(-3\alpha).(-\alpha).\ \alpha\ .\ 3\alpha.\ 5\alpha = -225$.

   Thus $\alpha^6 = 1$,
   so $\alpha = 1$.

   The six roots are therefore $-5, -3, -1, 1, 3, 5$. The given equation can be written in the form $(x^2 - 1)(x^2 - 9)(x^2 - 25) = 0$, from which we get $p = -35$, $q = 259$.

5.



$$S = (0, \eta), \quad \eta = \pm\sqrt{1 - \frac{1}{4}c^2} .$$

R lies on circle with centre S and radius 1:

$$x^2 + (y - \eta)^2 = 1$$

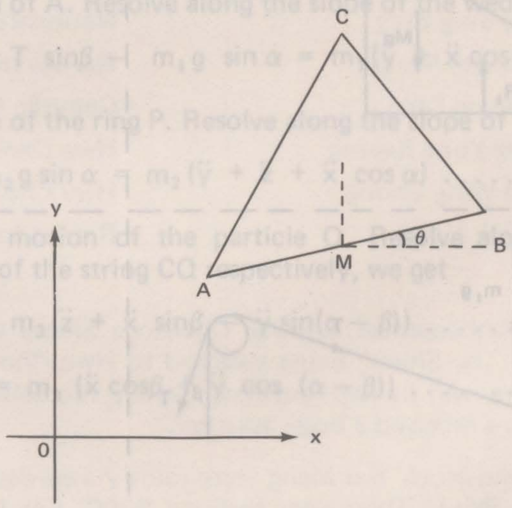
$$x^2 + y^2 - 2\eta y = 1 - \eta^2 = \frac{1}{4}c^2.$$

$$\overline{QR}^2 = (x - \frac{1}{2}c)^2 + y^2$$
$$= x^2 + y^2 - cx + \frac{1}{4}c^2$$
$$= \frac{1}{2}c^2 + 2\eta y - cx$$

$$\overline{RP}^2 = (x + \frac{1}{2}c)^2 + y^2$$
$$= \frac{1}{2}c^2 + 2\eta y + cx$$

$$\overline{QR}^2 \,.\, \overline{RP}^2 = (\frac{1}{2}c^2 + 2\eta y)^2 - c^2x^2$$
$$= \frac{1}{4}c^4 + 2c^2\eta y + 4\eta^2y^2 - c^2x^2$$
$$= \frac{1}{4}c^4 + 2c^2\eta y + 4y^2\ (1 - \frac{1}{4}c^2) - c^2x^2$$
$$= \frac{1}{4}c^4 + 2c^2\eta y + 4y^2 - c^2(x^2 + y^2)$$
$$= \frac{1}{4}c^4 + 2c^2\eta y + 4y^2 - c^2(2\eta y + \frac{1}{4}c^2)$$
$$= 4y^2$$

Alternatively, let $\widehat{PRQ} = \alpha$. Area of $\Delta PQR = \frac{1}{2}\ \overline{PR}\ .\ \overline{RQ}\ \sin\alpha = \frac{1}{2}c\ |y|$. But $\widehat{PSQ} = 2\alpha$, where $\alpha = \widehat{PRQ}$, and so $\widehat{PSO} = \widehat{OSQ} = \alpha$, and $\sin\alpha = \frac{c}{2}$. We get $\overline{PR}\ .\ \overline{RQ} = 2\,|y|$, so $\overline{PR}^2, \overline{RQ}^2 = 4y^2$.



Let AB make an angle θ with the x-axis. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $M = (h, k)$, where $h = \frac{1}{2}(a_1 + b_1)$, $k = \frac{1}{2}(a_2 + b_2)$. Perform a change of coordinates given by a translation of the origin O to M, followed by a rotation about M through an angle θ :

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x-h \\ y-k \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x-h \\ y-k \end{pmatrix}$$

Let the coordinates of C with respect to the new axes be (c_1', c_2').

Then $c_2' = (c_1 - h)\sin\theta + (c_2 - k)\cos\theta$.

In view of the first part, we have $a^2b^2c^2 = 4c^2 c_2'^2$

$$= 4\,[(c_1 - h)(a_2 - b_2) + (c_2 - k)(a_1 - b_1)]^2$$

$$= 4\,[a_1(c_2 - a_2) + b_1(b_2 - c_2) + c_1(a_2 - b_2)]^2,$$

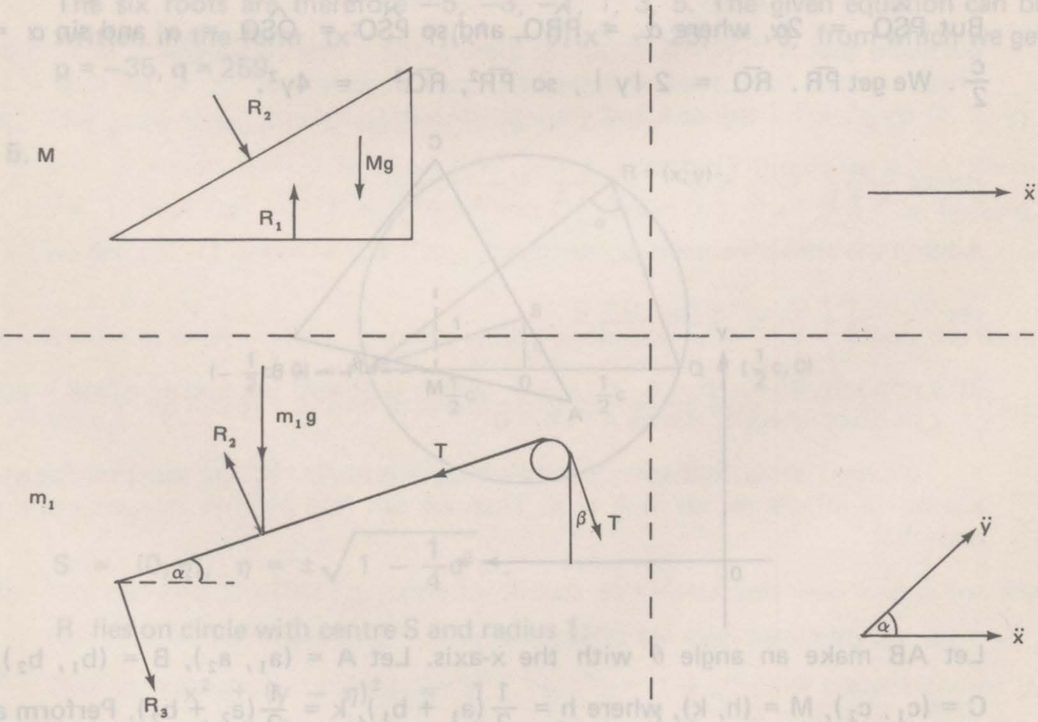
where we have substituted for h, k and

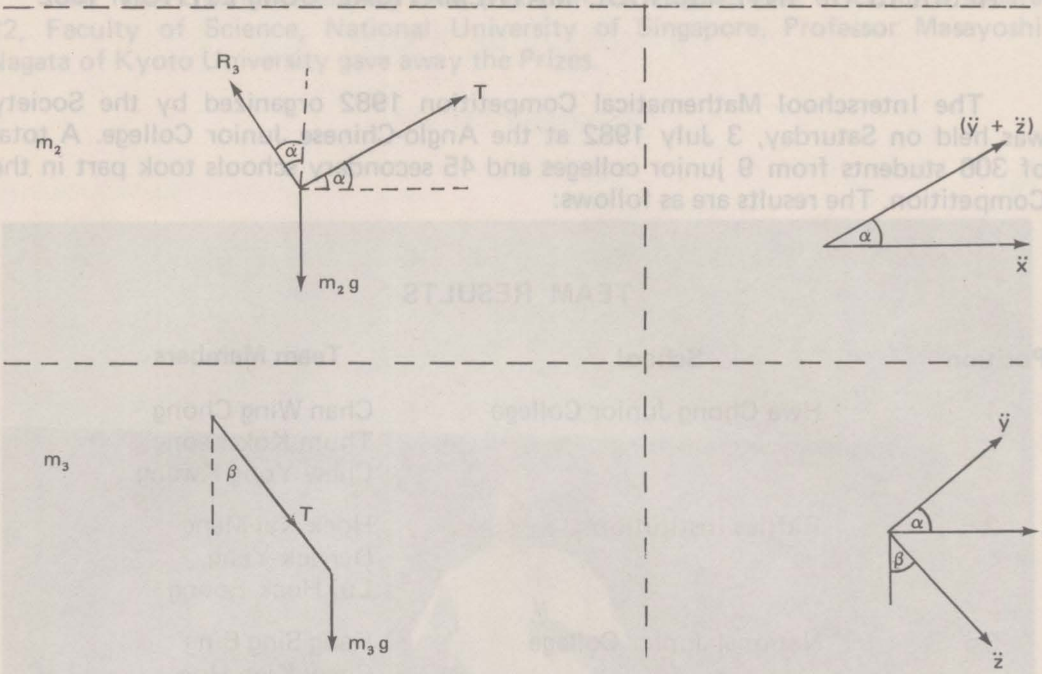
$$\sin\theta = \frac{a_2 - b_2}{c}, \quad \cos\theta = \frac{a_1 - b_1}{c}.$$

Since $a_1, a_2, b_1, b_2, c_1, c_2$ are integers and $a^2b^2c^2 \neq 0$, it follows that $a^2b^2c^2 \geqslant 4$.

6. The forces acting on masses M, m_1, m_2, m_3 respectively and the respective accelerations relative to earth are as shown in the following diagrams. (R_1, R_2 and R_3 are reactions).

Forces **Accelerations relative to earth**





Consider motion of the whole system. Resolve forces horizontally, we get

$$0 = M\ddot{x} + m_1(\ddot{x} + \ddot{y}\cos\alpha) + m_2(\ddot{x} + (\ddot{y} + \ddot{z})\cos\alpha) + m_3(\ddot{x} + \ddot{y}\cos\alpha + \ddot{z}\sin\beta) \quad \ldots\ldots (1)$$

Consider motion of A. Resolve along the slope of the wedge:

$$-T + T\sin\beta - m_1 g \sin\alpha = m_1(\ddot{y} + \ddot{x}\cos\alpha) \quad \ldots\ldots (2)$$

Consider motion of the ring P. Resolve along the slope of the wedge:

$$T - m_2 g \sin\alpha = m_2(\ddot{y} + \ddot{z} + \ddot{x}\cos\alpha) \quad \ldots\ldots (3)$$

Lastly consider motion of the particle Q. Resolve along and perpendicular to the direction of the string CQ respectively, we get

$$-T = m_3(\ddot{z} + \ddot{x}\sin\beta - \ddot{y}\sin(\alpha - \beta)) \quad \ldots\ldots (4)$$

and $-m_3 g \sin\beta = m_3(\ddot{x}\cos\beta + \ddot{y}\cos(\alpha - \beta)) \quad \ldots\ldots (5)$