SOME REMARKS ON THE STUDY OF MATHEMATICS*

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It is my pleasure to give a talk at this ceremony, and I like to begin with a remark on the nature of understanding something. You know the equality $(-a) \times (-b) = ab$. Just before I learned this equality, my teacher explained to me negative numbers in comparison with owing money and borrowing money, and the explanation was understandable. But, coming to multiplication, I was amazed. "What is -1 times borrowing money?" The teacher said, "If we multiply -1 to numbers, then positive numbers turn to negative numbers. Therefore negative numbers should turn to positive numbers." I was not convinced by this explanation. Later, I noticed that $(a-b)(c-d) = ac + bd - ad - bc$, at least in the case $a > b$ and $c > d$. Then I felt that I understood the equality $(-a) \times (-b) = ab$. As another example, when I learned the notion of vectors, though I could follow calculation of vectors, I felt that vectors are something out of my understanding. I could not see the meaning of equality or calculation of vectors. At that time, I could explain the notion of vectors by a sentence. I knew about vectors just as a knowledge, but I could not understand the notion. Later, faced with several applications of vectors, I began to feel that I understood vectors.

I suppose that many of you have some experience like this. Even if you once felt that you understood something, you may feel later that you did not really understand it, because your understanding has become deeper. In any case, in going ahead, it is important to bring you to a situation in which you feel that you understand the notion, theorem, formula, etc.

When you come to a new notion, etc., if you have enough understanding about the circumstances, then you can understand it quite easily; otherwise, you will understand it after some exercises or after getting some information connected with it. This is one reason why solving exercises is important in studying mathematics. But, it is also important to try to understand a mathematical subject from various sides.

What I like to speak about next is the question of who could be a mathematician. There are many people who believe that mathematicians are skilled in calculation of numbers. This belief is wrong. If an engineer designing an air plane makes an error, then there may be some fatal accidents. In general, engineers should have a good ability in calculation. Errors made by a mathematician could hardly be fatal. Furthermore, it is not often that a mathematician does complicated calculation of numbers. Thus ability in calculation of numbers is not important to

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Biographical Note. Professor Masayoshi Nagata is Professor of Mathematics at Kyoto University in Japan and has been a visiting professor or researcher at many universities, including Harvard, Northwestern, Purdue and the University of Cambridge. His primary research interests are in commutative algebra and algebraic geometry. Among his many contributions to mathematics is his solution of Hilbert's 14th Problem. He was a Vice-President of the International Mathematical Union.

becoming a mathematician. To be a good mathematician the most important ability is the ability to think of problems from various sides. This includes the ability to create good new problems, to clarify the circumstances and to try to understand the circumstances deeper and deeper.

Without any interest in a problem, we cannot continue to think about the problem. Therefore, for this kind of ability, you must promote your interest in mathematics. Thus, my answer to the question, "Who could be a mathematician?" is that anybody could be a mathematician if he promotes his interest in mathematics. For this purpose, I like to advise you to try the following practice:

(I) When you see a mathematical problem somewhere and if you cannot see how to solve it, then write down the problem on a sheet of paper as simply as possible. Keep the sheet in your pocket. If you find some leisure time, take out the sheet, recall the problem and try to solve it. As soon as you can see how to solve it, stop going further into the problem, and pick up the next problem. If time is up and you do not see how to solve it, then keep the sheet in your pocket again.

(II) Try to modify problems and theorems. Namely, think of what it would be if you change the assumptions. One easy example of this kind is: You know that if a circle and a line are given on a plane, then (i) they cross at two points or (ii) the line is tangent to the circle or (iii) they have no common point. You may think what it would be if you replace the circle by a polygon or by a sphere in space. For any theorem, there can be many such modifications; some useful, some nonsense. Anyway, you should try.

Now I like to speak about imagination. In advancing mathematics, good imagination is very important. Mathematics is very abstract. To be abstract means to be more applicable in a sense. But, abstract notions are usually hard to understand and it is also hard to have motivation to develop them further. This is why imagination is important in advancing mathematics. Thus we can say that most of mathematics are products of imagination. I shall speak about three such examples.

The first example is the notion of negative numbers. If one handles addition and subtraction only, then negative numbers are understood rather easily, though one needs some imagination. For multiplication, one needs good imagination; one has to think of an abstract notion of numbers.

Though negative numbers were recognized in ancient China, as they appeared in an old Chinese book 九章算術, in Europe, negative numbers were recognized as numbers only in the 16th century.

The second is the notion of complex numbers. It was also in the 16th century that Cardano used square roots of negative numbers to compute roots of algebraic equations of degree three. It is believed that the one who found a method to solve equations of degree three is Scipione del Ferro, who did not publish his solution. But, in those days, negative numbers were not recognized and therefore we may say that he found a method to find one solution of a certain type of equations of degree three. Cardano used not only negative numbers but also square roots of negative numbers. He applied a similar method to the general case to obtain three roots. Though you probably do not know the method of Cardano, I like to tell you the fact that if all three roots are mutually distinct real numbers, then we have to use square roots of negative numbers in solving the equation. Cardano did not give a rigorous foundation of complex numbers. He used imaginary numbers just for convenience. The foundation was given much later, and in the view of modern mathematicians, the existence of complex numbers is real. The introduction of the notion of complex numbers was the good imagination of Cardano.

I suppose that you have heard of non-Euclidean geometry. This is a geometry in which the usual axiom of parallel lines is denied. In that sense, there are two types of geometry. One type is such that if a line ℓ and a point p outside of ℓ are given, there are at least two lines going through p and parallel to ℓ. The other type is such that there is no line going through p and parallel to ℓ. In the old days, many people imagined that the axiom of parallel lines could be proved and they tried to prove it. At last, people noticed that we can have non-Euclidean geometry and found several models satisfying one of these axioms of plural existence or non-existence of parallel lines. At that time, the study was done merely from the logical view-point. But the existence of such a theory helped Einstein when he proposed relativity theory. The structure of our universe is still mysterious and is seemingly a non-Euclidean space.

As I said before, there are plenty examples of products of good imagination. But I will not give more examples here. Instead, for your better understanding, I like to say something more about complex numbers and non-Euclidean geometry.

Though you may know about the Gauss plane, let us review it. Consider a plane with coordinate system (x, y). The point (x, y) represents the complex number $x + iy$ ($i = \sqrt{-1}$). Addition is coordinatewise. Therefore complex numbers have the property of two-dimensional vectors. Furthermore, complex numbers admit multiplication. For multiplication, it is better to use another expression by absolute value and argument. Namely, take a complex number α on the Gauss plane with origin O. The length of $\overrightarrow{O\alpha}$ is the absolute value of α, usually denoted by $|\alpha|$. The angle θ between $\overrightarrow{01}$ and $\overrightarrow{O\alpha}$ is the argument of α, usually denoted by Arg α. Then α is expressed by

$$\alpha = r(\cos\theta + i \cdot \sin\theta),$$

with $r = |\alpha|$ and $\theta = \text{Arg}\,\alpha$. Now, if $\beta = s(\cos\ell + i{\cdot}\sin\ell)$, where $s = |\beta|$, $\ell = \text{Arg}\,\beta$, then $\alpha\beta = rs(\cos(\theta + \ell) + i{\cdot}\sin(\theta + \ell))$. Thus

$$|\alpha\beta| = |\alpha|\,|\beta|, \quad \text{Arg}\,(\alpha\beta) = \text{Arg}\,\alpha + \text{Arg}\,\beta \text{ (modulo } 2\pi).$$

So, for instance, to multiply a complex number of absolute value 1 means a rotation of the Gauss plane about O.

This kind of property of complex numbers is very often used in mathematics, and I like to add one rather odd application of the Gauss plane which may interest you.

Take two points A, B ($A \neq B$) on a plane. For a positive number r, the set $C_r = \{P \mid AP/BP = r\}$ forms a circle known under the name of Appollonius if $r \neq 1$; if $r = 1$, then C_r is a line. Now there is a theorem:

If D is a circle going through both A and B, then C_r and D intersect orthogonally.

This theorem can be proved easily by using the mapping f of the Gauss plane G without the origin, defined by $f(z) = z^{-1}$. This mapping has the following properties:

(1) lines going through O $\overset{f}{\leftrightarrow}$ lines going through O.

(I) (2) circles not going through O $\overset{f}{\leftrightarrow}$ circles not going through O.

(3) lines not going through O $\overset{f}{\leftrightarrow}$ circles not going through O.

(II) If ϵ and δ approach α and if the limit of the angle between $\epsilon\alpha$ and $\delta\alpha$ is θ, then $f(\epsilon)$ and $f(\delta)$ approach $f(\alpha)$ and the limit of the angle between $\overrightarrow{f(\epsilon)f(\alpha)}$ and $\overrightarrow{f(\delta)f(\alpha)}$ is equal to θ. This means that two curves intersect at α at an angle θ if and only if the images of these curves intersect at α^{-1} at an angle θ.

These properties are easily confirmed and I shall leave the details to you.

Let us apply this to our theorem. We may let A = 0, B = 1. Apply f to D : f(D) is a line going through 1. For $r \neq 1$, C_r is a circle symmetric with respect to the real line. Therefore $f(C_r)$ is a circle symmetric with respect to the real line. The real points of C_r are $r/(1 + r)$ and $r/(r - 1)$. Therefore $f(C_r)$ is the circle having $(1 + r)/r$ and $(r - 1)/r$ as ends of a diameter. Hence $f(C_r)$ is a circle with center at 1. Therefore f(D) and $f(C_r)$ intersect orthogonally. The case r = 1 is clear. Thus by (II) the theorem is proved.

In ending this lecture, I like to introduce the projective plane as an easy model of non-Euclidean geometry.

Roughly speaking, we first consider the usual plane, say F, and we add so-called points at infinity so that two lines on F meet at infinity if and only if they are parallel on F. This can be realized by using coordinate system (x, y, z) as follows:

We consider the set $H = \{ (x, y, z) \mid (x, y, z) \neq (0, 0, 0) \}$. Two coordinates (a, b, c) and (a′, b′, c′) express the same point if and only if a:b:c = a′:b′:c′. If we associate (x, y, 1), or equivalently (xt, yt, t), $t \neq 0$, to the point (x, y) on F, then we have a good embedding of F in the projective plane and those having coordinates of type (a, b, 0) are points at infinity. Lines are defined to be the set of points satisfying a linear equation of the type aX + bY + cZ = 0, $(a, b, c) \neq (0, 0, 0)$. A line on F is given by aX + bY + c = 0. This is embedded in the line aX + bY + cZ = 0, which has (−b, a, 0) as a unique extra point, the point at infinity which is to be added to the line aX + bY + c = 0. The projective plane has only one line, Z = 0, outside of F. Therefore any two lines in the projective plane have a common point.

In the projective plane, we can observe some interesting phenomena. For instance, circles, ellipses, parabolas, and hyperbolas are transformed by linear transformations into each other. This matches with the fact that they are conic sections.

Another interesting phenomenon is duality. Consider the correspondence:

$$\text{point } (a, b, c) \leftrightarrow \text{line } aX + bY + cZ = 0.$$

A point (a', b', c') lies on the line $aX + bY + cZ = 0 \leftrightarrow aa' + bb' + cc' = 0 \leftrightarrow$ the line $a'X + b'Y + c'Z = 0$ goes through the point (a, b, c).

Therefore, if we have a statement on lines and points with respect to "lying on" and "going through", then the new statement obtained by interchanging "point" with "line" and "lying on" with "going through" is equivalent to the original statement.

I shall give you one example by proving the following theorem:

Assume that a given finite set M of points $p_1, \ldots, p_n$ in space has the property that if a line goes through two of these p_i, then it goes through at least three of these p_i. Then $p_1, \ldots, p_n$ are collinear.

Proof. It is sufficient to prove the case when these p_i are on a plane. Therefore it suffices to prove this on a projective plane. Then we have only to prove the dual:

(*) Assume that a given finite set M^* of lines $\ell_1, \ldots, \ell_n$ has the property that if a point p is common to two of these ℓ_i, then there are at least three of these ℓ_i which go through p. Then all the ℓ_i go through one common point.

We take a line ℓ^* not going through any of the intersections of these lines. Then the complement of ℓ^* has the structure of the usual plane. Therefore it sufficies to prove (*) in F under additional assumption that each pair ℓ_i, ℓ_j has a common point.

Assume that there are more than one intersection. Take ℓ_1. There is an intersection outside of ℓ_1. Take the nearest intersection outside of ℓ_1. Let it be p. Through p, there are at least three of the ℓ_i, say ℓ_2, ℓ_3, ℓ_4. Set $q_i = \ell_i \cap \ell_1$, $(i = 2, 3, 4)$, and we may assume that q_2 is in between q_3 and q_4. Then through q_2, there must be one ℓ_i other than ℓ_1 and ℓ_2. Let it be ℓ_5. Then ℓ_5 has an intersection with either ℓ_3 or ℓ_4, nearer to ℓ_1 than p. This is a contradiction, and the theorem is proved.