THE WILBRAHAM – GIBBS PHENOMENON IN FOURIER ANALYSIS*

C. W. Onneweer
University of New Mexico
Albuquerque NM 87131, USA

The Gibbs phenomenon in Fourier analysis deals with the behavior of the partial sums of the Fourier series of certain functions near a point of discontinuity of these functions. In this paper we will explain this behavior and give some information about the history of the Gibbs phenomenon.

1. Fourier series

Throughout this paper we shall consider functions f defined on the real line that are periodic with period 2π, i.e. $f(x + 2\pi) = f(x)$ for all x, and that are piecewise continuous. This last condition means that on each finite interval f is continuous except at at most finitely many points where we only require that the left-and right-sided limits of the function exist.

In the theory of Fourier series we try to express such a 2π-periodic function as an infinite sum of the "basic" 2π-periodic functions sin nx, cos nx or e^{inx}. Disregarding, at least for the moment, questions of convergence, we can associate with a given function f two infinite series

$$(1)\quad f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

$$(2)\quad f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx},$$

where

$$(3)\quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\, dt, \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\, dt,$$

$$(4)\quad \hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{ikt} dt.$$

The series defined in (1) and (2) are called the real or complex Fourier series of f and the numbers a_k, b_k and $\hat{f}(k)$ defined in (3) and (4) are called the real and complex Fourier coefficients of f. Since for each integer k we have $e^{ikx} = \cos kx + i \sin kx$, it is clear that the a_k and b_k can be expressed in terms of the $\hat{f}(k)$ and vice versa. To explain why the complex Fourier coefficients are defined as in (4) assume, for a moment, that f(x) can be expressed as the sum of an infinite series of the form

$$(5)\quad f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

*Text of an invited lecture delivered to the Society on 21 October 1982 at the National University of Singapore.

If, moreover, f is so smooth that the series converges uniformly to f(x) then we have for an arbitrary integer k_o

$$f(x)\, e^{-ik_o x} = \sum_{k=-\infty}^{\infty} c_k e^{i(k-k_o)x}.$$

Integrating both sides from 0 to 2π and interchanging summation and integration, we see that

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ik_o x}\,dx = \sum_{k=-\infty}^{\infty} c_k \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-k_o)x}\,dx = c_{k_o},$$

because

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(k-k_o)x}\,dx = \begin{cases} 0 & \text{if } k \neq k_o, \\ 1 & \text{if } k = k_o. \end{cases}$$

Thus, for sufficiently smooth functions, the only way to express f as an infinite series like in (5) is to choose $c_k = \hat{f}(k)$ as defined in (4).

In our first theorem we shall give a sufficient condition that implies that the complex Fourier series of a given function converges to the function at some point. However, we first prove a simple lemma for complex Fourier coefficients.

Lemma 1. (a) For each integer k we have $|\hat{f}(k)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\,dx.$

(b) $\lim_{k\to-\infty} \hat{f}(k) = \lim_{k\to\infty} \hat{f}(k) = 0.$

Proof. Part (a) is trivial. To prove (b), first assume that f is continuous. For each k we have

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt$$

$$= \frac{1}{2\pi}\int_{\frac{\pi}{k}}^{2\pi+\frac{\pi}{k}} f(t-\tfrac{\pi}{k})\, e^{-ik(t-\frac{\pi}{k})}\,dt.$$

i.e., $$\hat{f}(k) = -\frac{1}{2\pi}\int_0^{2\pi} f(t-\tfrac{\pi}{k})e^{-ikt}\,dt.$$

Therefore,

$$|2\hat{f}(k)| = \left|\frac{1}{2\pi}\int_0^{2\pi} (f(t) - f(t-\tfrac{\pi}{k}))e^{-ikt}\,dt\right|$$

$$\leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(t) - f(t-\tfrac{\pi}{k})|\,dt$$

$$\leqslant \max\{|f(t) - f(t-\tfrac{\pi}{k})| \,;\, t \in [0,2\pi]\}.$$

Thus $|\hat{f}(k)| \to 0$ as $k \to \pm\infty$, because, actually, f is uniformly continuous on $[0, 2\pi]$. If f is only piecewise continuous, choose, for given $\epsilon > 0$, a continuous 2π-periodic function g such that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t) - g(t)|\, dt < \epsilon.$$

Then $|\hat{f}(k)| \leqslant |\hat{f}(k) - \hat{g}(k)| + |\hat{g}(k)|$, which, combined with part (a), immediately implies (b).

Theorem 1. If $f'(x)$ exists in a neighborhood of a point x_o, then the complex Fourier series of f converges at x_o to $f(x_o)$.

Proof. Replacing, if necessary, $f(x)$ by $f(x + x_o) - f(x_o)$, we may assume, without loss of generality, that $x_o = 0$ and $f(x_o) = 0$. Define the function g by

$$g(x) = \begin{cases} (e^{ix} - 1)^{-1} f(x) & \text{if } x \neq n\cdot 2\pi, \\ -i f'(0) & \text{if } x = n\cdot 2\pi. \end{cases}$$

Then g is continuous at $x = 0$ (ℓ 'Hôpital's Rule) and, consequently, g is piecewise continuous and 2π-periodic. Since $f(x) = (e^{ix} - 1)g(x)$ for $x \neq n\cdot 2\pi$, we see that $\hat{f}(k) = \hat{g}(k-1) - \hat{g}(k)$ for each integer k. Therefore,

$$\sum_{k=-n}^{n} \hat{f}(k)e^{ik\cdot 0} = \sum_{k=-n}^{n} (\hat{g}(k-1) - \hat{g}(k))$$

$$= \hat{g}(-n-1) - \hat{g}(n) \to 0 = f(0)$$

as $n \to \infty$, according to Lemma 1(b). This completes the proof of the theorem.

Remark. (i) Lemma 1(b) is known as the Riemann-Lebesgue Lemma.

(ii) The proof of Theorem 1 as given here is due to P. Chernoff, see [1] for additional details.

(iii) Lemma 1 and Theorem 1 also hold for real Fourier coefficients and Fourier series.

2. The Gibbs phenomenon

We begin this section by giving some examples of piecewise continuous, 2π-periodic functions and their real Fourier series. It may help the reader to sketch the graph of each of these functions and to compute at least some of the Fourier coefficients directly from the definition given in (3). As we shall see in Section 3, the first three examples all played a role in the history of the Gibbs phenomenon.

(6) If $f(x) = \frac{1}{2}(\pi - x)$ for $0 < x < 2\pi$ and $f(0) = 0$, then $f(x) = \sum_{k=1}^{\infty} \frac{1}{k}\sin kx$.

(7) If $g(x) = \frac{1}{2}x$ for $-\pi < x < \pi$ and $g(\pi) = 0$, then $g(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx$.

(8) If $h(x) = \frac{\pi}{4}$ for $|x| < \frac{\pi}{2}$, $h(x) = -\frac{\pi}{4}$, $\frac{\pi}{2} < |x| < \pi$, $h(\pm\frac{\pi}{2}) = 0$, then

$$h(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \cos(2k-1)x.$$

(9) If $\phi(x) = -\frac{\pi}{2}$ for $-\pi < x < 0$, $\phi(x) = \frac{\pi}{2}$, $0 < x < \pi$, $\phi(0) = \phi(\pi) = 0$, then

$$\phi(x) = 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(2k-1)x.$$

Note that, according to the Theorem 1, the Fourier series of each of these functions converges to the function at the points of continuity and, clearly, the same is true at the points of discontinuity. As we shall see, the partial sums of the Fourier series show the same kind of peculiar behavior near the points of discontinuity of the corresponding limit functions. We shall first consider this behavior for the function $\phi(x)$ of example (9). Define ϕ_n by

$$\phi_n(x) = 2 \sum_{k=1}^{n} \frac{1}{2k-1} \sin(2k-1)x.$$

Then ϕ_n is 2π-periodic, $\phi_n(-x) = -\phi_n(x)$ and $\phi_n(0) = \phi_n(\pi) = 0$. So, we only need to study ϕ_n for $x \in (0, \pi)$. To find the local maxima and minima of ϕ_n we consider

$$\begin{aligned}\phi_n'(x) &= 2 \sum_{k=1}^{n} \cos(2k-1)x \\ &= 2(\sin x)^{-1} \sum_{k=1}^{n} \cos(2k-1)x \sin x \\ &= 2(\sin x)^{-1} \sum_{k=1}^{n} \frac{1}{2}(\sin 2kx - \sin(2k-2)x) \\ &= (\sin x)^{-1} \sin 2nx.\end{aligned}$$

Thus,

$$\phi_n(x) = \int_0^x \frac{\sin 2nt}{\sin t}\, dt \tag{10}$$

and $\phi_n'(x) = 0$ on $(o, \pi) \Leftrightarrow x = m\frac{\pi}{2n}$, $1 \leqslant m < 2n$.

The following results are of interest to us.

Theorem 2. (a) $\phi_n(x)$ has its relative maxima on $(0, \pi)$ at $x = (2\ell - 1)\frac{\pi}{2n}$ $(1 \leqslant \ell \leqslant n)$ and $\phi_n(x)$ has its relative minima on $(0, \pi)$ at $x = 2\ell\frac{\pi}{2n}$ $(1 \leqslant \ell < n)$.

(b) $\phi_n((2\ell-1)\frac{\pi}{2n}) > \phi_n((2\ell+1)\frac{\pi}{2n})$ $(1 \leqslant \ell < \frac{n}{2})$.

(c) $\max\{\phi_n(x);\ x \in (0,\pi)\} = \phi_n(\frac{\pi}{2n})$.

(d) $\phi_{n+1}(\frac{\pi}{2(n+1)}) > \phi_n(\frac{\pi}{2n})$.

(e) $\lim\limits_{n\to\infty} \phi_n(\frac{\pi}{2n}) = \int_0^{\pi} \frac{\sin t}{t}\, dt = S.$

Proof. (a) follows immediately from a computation of $\phi_n''(x)$ at $x = m\frac{\pi}{2n}$. (b) follows easily from the representation of $\phi_n(x)$ as given in (10) and (c) is a direct consequence of (b). To prove (d), notice that (c) implies

$$\phi_{n+1}(\frac{\pi}{2(n+1)}) \geqslant \phi_{n+1}(\frac{\pi}{2n})$$

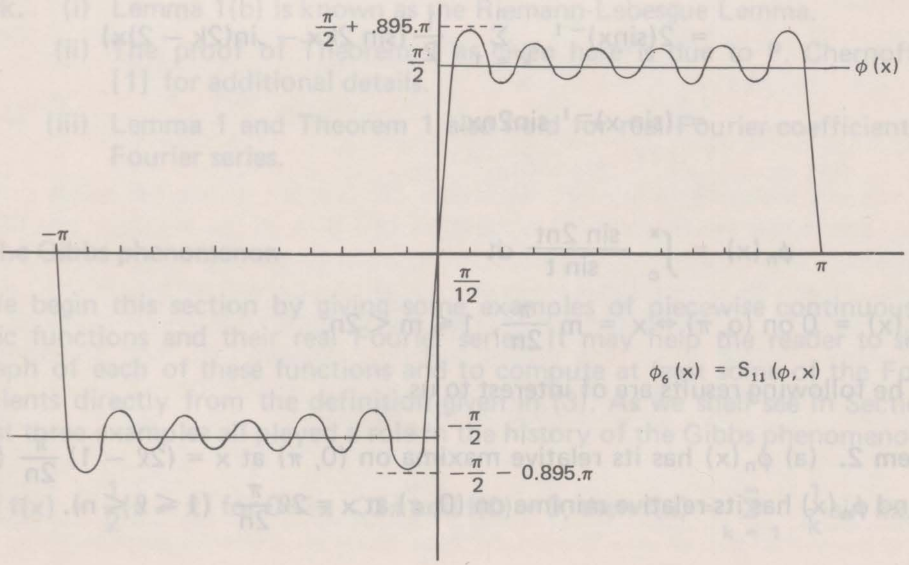
$$= \phi_n(\frac{\pi}{2n}) + \frac{2}{2n+1}\sin(2n+1)\frac{\pi}{2n} > \phi_n(\frac{\pi}{2n}).$$

Finally, to prove (e) we observe that

$$\phi_n(\frac{\pi}{2n}) = 2\sum_{k=1}^{n} \frac{1}{2k-1}\sin(2k-1)\frac{\pi}{2n}$$

$$= \sum_{k=1}^{n} \frac{\sin(2k-1)\frac{\pi}{2n}}{(2k-1)\frac{\pi}{2n}} \cdot \frac{\pi}{n},$$

which is a Riemann sum for the integral in (e) corresponding to the partition of $(0,\pi)$ into n equal intervals and taking the value of the integrand $t^{-1}\sin t$ at the midpoint of each of these intervals. This completes the proof of (e).



In view of Theorem 2(e) it becomes important to find the value of S. According to [2] we have

$$\lim_{n \to \infty} \phi_n(\frac{\pi}{2n}) = S = 1.8519370\ldots$$
$$= .5895\ldots\cdot\pi$$
$$= .5895\ldots\ldots(\phi(0^+) - \phi(0^-))$$
$$= \frac{\pi}{2} + .0859\ldots\ldots(\phi(0^+) - \phi(0^-)).$$

Thus the functions $\phi_n(x)$, which are partial sums of the Fourier series of ϕ, exceed or overshoot the function ϕ near its point of discontinuity $x = 0$ by approximately 9% of the jump in ϕ-values at $x = 0$. More formally, we have

$$\lim_{n \to \infty} \left(\max_{x \in (0,\pi)} \phi_n(x) \right) \neq \max_{x \in (0,\pi)} \left(\lim_{n \to \infty} \phi_n(x) \right).$$

This overshoot is an example of the so-called Gibbs phenomenon for Fourier series. As will be shown in the next theorem, a similar overshoot is exhibited by many functions. To simplify our notation, let $S_n(f\,;x)$ denote the n-th partial sum of the Fourier series of a function f, i.e., $S_n(f\,;x) = \frac{a_o}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$. Note that with this notation, $\phi_n(x) = S_{2n-1}(\phi\,;x)$.

Theorem 3. Let g be a real-valued 2π-periodic function so that g and g′ are both piecewise continuous. Assume that g has a discontinuity at $x = a$ and that $g(a^+) - g(a^-) = \alpha$. Then the partial sums of the Fourier series of g overshoot the function g in a neighborhood of $x = a$ by approximately 9% of the size of the jump α.

Proof. (OUTLINE). First define $\tilde{g}$ by

$$\tilde{g}(x) = g(x+a) - \frac{1}{2}(g(a^+) + g(a^-)).$$

Then $\tilde{g}$ has a discontinuity at $x = 0$ and $\tilde{g}(0^-) = -\tilde{g}(0^+)$. Next define the function h by

$$h(x) = \begin{cases} \tilde{g}(x) - \frac{\alpha}{\pi}\phi(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then h is continuous at $x = 0$ and both h and h′ are piecewise continuous. According to a theorem due to Jordan, the Fourier series of h converges uniformly to h(x) in a neighborhood of $x = 0$. Therefore,

$$\lim_{n \to \infty} S_{2n-1}(h\,;\frac{\pi}{2n}) = h(0) = 0.$$

Consequently,

$$\lim_{n \to \infty} S_{2n-1}(\tilde{g}; \frac{\pi}{2n}) = \frac{\alpha}{\pi} \lim_{n \to \infty} S_{2n-1}(\phi; \frac{\pi}{2n})$$

$$= \frac{\alpha}{\pi} \lim_{n \to \infty} \phi_n(\frac{\pi}{2n})$$

$$= \frac{\alpha}{\pi} \cdot S = \frac{S}{\pi}\alpha = .5895 \ldots \alpha.$$

Since $S_{2n-1}(g; a + \frac{\pi}{2n}) = S_{2n-1}(\tilde{g}; \frac{\pi}{2n}) + \frac{1}{2}(g(a^+) + g(a^-))$, we see that

$$\lim_{n \to \infty} S_{2n-1}(g; a + \frac{\pi}{2n}) = \frac{1}{2}(g(a^+) + g(a^-)) + .5895 \ldots \alpha.$$

Comparing this formula with formula (11) we see that the partial sums of the Fourier series of g near x = a display the same behavior as the partial sums of the Fourier series of ϕ near x = 0. This peculiar behavior is called the Gibbs phenomenon.

3. Historical remarks

The following remarks are extracted from the fascinating article by E. Hewitt and R. E. Hewitt [2]. Their paper contains a wealth of information on the history of the Gibbs phenomenon. It also contains a much more detailed analysis of this phenomenon than is given here. Their article is highly recommended and is a pleasure to read.

In October 1898 the physicist A. A. Michelson wrote a letter to Nature, in which he criticized "the idea that a real discontinuity can replace a sum of continuous curves", that is to say, the idea that a series of continuous functions could converge to a discontinuous function. Michelson used the series given in (7) to "explain" his contention but he was clearly confused about the difference between the sum of an infinite series and the (large) partial sums of such a series. In the next issue of Nature the mathematician A. E. H. Love gave a rather sarcastic response to Michelson's letter, in which he pointed out Michelson's confusion and lack of understanding. One month later, J. W. Gibbs joined the discussion in Nature. In his first letter he tried to clarify Love's explanation, stating that the limit of the graphs (of the partial sums of a Fourier series) is not necessarily the same as the graph of the limit. In a second letter published in Nature in April 1899, Gibbs mentioned the quantity $S = \int_0^\pi t^{-1} \sin t \, dt$, which determines the amount of overshoot as explained in §2 of this paper. It is of some interest to mention here that Gibbs gave no hint of a proof for any of his assertions.

In 1906 M. Bôcher published a long paper in which he gave a detailed analysis of the behavior of the Fourier series of the function given in (6). He also proved, essentially, our Theorem 3 and he introduced the terminology "Gibbs phenomenon".

In 1912 T. H. Gronwall published a much more detailed analysis of the Fourier series (6). He gave a wealth of new results, in particular, he proved several interesting properties for the other relative maxima and minima of the partial sums $S_n(g\ ;\ x)$. Although Gronwall, in passing, mentions Bôcher's paper, the latter in a 1914 paper severely criticized Gronwall for not giving him enough credit for his work of 1906. In a reaction to Bôcher's paper, L. Fejér noted that Bôcher's claims to priority were by and large unjustified and that only "after the publication of Herr T. H. Gronwall (1912) certain questions can in fact today be handled with the greatest ease, for which, however, in the year 1906 every trace of a hint was lacking".

Then towards the end of 1914 H. Burkhardt described some long-forgotten work by the British mathematician H. Wilbraham, who already in 1848 had discovered Gibbs's phenomenon when he studied the Fourier series of the function h of example (8). Thus it would seem to be more appropriate to call the phenomenon discussed here the Wilbraham-Gibbs phenomenon.

In short, even a cursory look at the history of the Wilbrahm-Gibbs phenomenon showed us some of the human side of the mathematical activity: we meet a forgotten pioneer in the person of Wilbraham, we saw some of the confusion about the meaning of convergence for infinite series that still existed around the turn of the century and we encountered a rather bitter dispute between Bôcher and Fejér about priorities of mathematical results. To conclude with a quote from [2] : "Gibbs's phenomenon and its history offer ample evidence that mathematics, for all its majesty and austere exactitude, is carried on by humans."

References

[1] P. Chernoff, *Pointwise convergence of Fourier series,* Amer. Math. Monthly, 87 (1980), 399-400.

[2] E. Hewitt & E. R. Hewitt, *The Gibbs-Wilbraham phenomenon: An episode in Fourier analysis,* Archive for History of Exact Sciences, 21 (1979), 129-160.