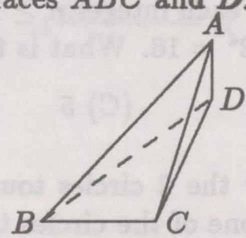
Singapore Mathematical Society
Interschool Mathematical Competition 1990
Part A

Saturday, 23 June 1990 — 1000–1100

Attempt as many questions as you can.
No calculators are allowed.
Circle your answers on the Answer Sheet provided.
Each question carries 5 marks.

1. Let $ABCD$ be a tetrahedron such that AD is perpendicular to the face BCD, $AC = 2$, $\angle ABC = 45°$, $\angle BAC = 15°$ and $\cos\theta = 1/\sqrt{3}$, where θ is the angle between the faces ABC and DBC. Then $\angle BDC =$

(A) 15°
(B) 20°
(C) 30°
(D) 45°
(E) None of the preceding



2. Let x, y, z, α, β be real numbers such that

$$\begin{aligned} x + y + 2z &= -2 \\ x + 2y + 3z &= \alpha \\ x + 3y + 4z &= \beta \\ x + 4y + 5z &= \beta^2 \end{aligned}$$

where α and β are not integers. Then $\alpha =$

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$ (C) $\frac{5}{4}$ (D) $-\frac{5}{4}$ (E) None of the preceding

3. Let f be a function such that

$$f(x + y^2) = f(x) + 2(f(y))^2 \quad \text{and} \quad f(1) \neq 0.$$

Find $f(1990)$.

(A) $\sqrt{1990}$ (B) 995 (C) 1990 (D) 2980 (E) $(1990)^2$

4. If 10 students, among whom are Anne and Bob, stand in a row at random, what is the most probable number of students standing between Anne and Bob?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

5. For each positive integer n, let $\sigma(n)$ be the number of positive divisors of n. For example $\sigma(4) = 3$ as 1, 2 and 4 are the positive divisors of 4. Find

$$\sum_{n=1}^{100} \left\{1 + (-1)^{\sigma(n)+1}\right\}.$$

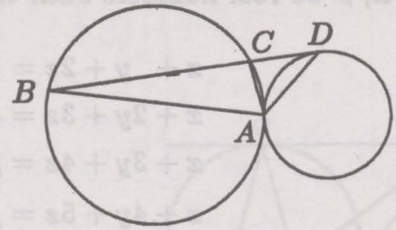
(A) 5 (B) 10 (C) 15 (D) 20 (E) 25

6. Let $u_1 = 2$, and for each integer $n \geq 1$, let $u_{n+1} = 2^{u_n}$. For example, $u_2 = 2^2 = 4$, $u_3 = 2^4 = 16$. What is the second last digit of u_{100} ?

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

7. In the figure below the 2 circles touch externally at A. The chord BC (extended) of one of the circles touches the other circle at D. If $\angle CBA = 12.5°$ and $\angle BAC = 67°$, then $\angle CAD =$

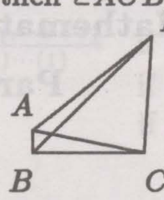
(A) 62.5°
(B) 60.5°
(C) 56°
(D) 60°
(E) 56.5°



8. The disc D is rolled inside the circle C. S is a point on D and the path it traces is shown. If the circumference of C is 100, what is the circumference of D?

(A) 45
(B) 50
(C) 55
(D) 60
(E) 65

9. In the following figure, $\angle ABC = 90°$ and BP bisects $\angle ABC$. If $\angle APC = 45°$ and $\angle APB = 2°$, then $\angle ACB =$



(A) 2°

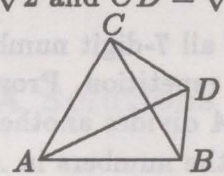
(B) 3°

(C) 4°

(D) 6°

(E) None of the preceding

10. Let D be a point outside an equilateral triangle ABC such that $\angle DBC = \angle DAC$. If $BD = \sqrt{2}$ and $CD = \sqrt{3}$, then $DA =$



(A) $\sqrt{2} + \sqrt{3}$

(B) $\sqrt{6}$

(C) $\sqrt{5}$

(D) $\sqrt[4]{6}$

(E) None of the preceding

—E N D—

Singapore Mathematical Society

Interschool Mathematical Competition 1990

Part B

Saturday, 23 June 1990 1100–1300

Attempt as many questions as you can.
No calculators are allowed.
Each question carries 25 marks.

1. Let A be the set of all 7-digit numbers formed using the digits 1, 2, 3, 4, 5, 6 and 7 without repetition. Prove that

(i) no number in A divides another;

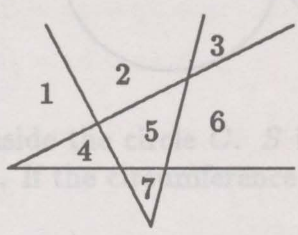
(ii) the sum of all the numbers in A is divisible by 9.

2. For each real number x, let $[x]$ denote the largest integer $\leq x$. For example $[2.3] = 2$, $[3] = 3$. Find all real numbers x that satisfy the equation

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = k$$

for (i) $k = 89$, (ii) $k = 90$.

3. Let Z_n be the maximum number of regions in the plane determined by n bent lines. For example, $Z_1 = 2$, $Z_2 = 7$ (see figure below). Prove that $Z_n = 2n^2 - n + 1$.



4. What is the value of $Q_{2^{100}}$ where

$$Q_n = \sum_{k=0}^{n} \binom{n-k}{k} (-1)^k \ ?$$

Note that for any real number r and any integer k

$$\binom{r}{k} = \begin{cases} \frac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots(1)} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k < 0 \end{cases}$$

5. Let P be a point inside ΔABC. Suppose $PA = \sqrt{2}$, $PB = 2$ and $PC = \sqrt{3} - 1$, find the maximum area of ΔABC.

—E N D—

Part A Solutions

1. Let E be the foot of the perpendicular from A to BC. Then $\angle AED = \theta$, $BE = AE = 2\sin 60° = \sqrt{3}$, $CE = 1$ and $DE = AE\cos\theta = 1$. So $\angle BDE = 60°$, whence $\angle BDC = \angle BDE - \angle CDE = 60° - 45° = 15°$.

2. We have $y+z = \alpha+2 = \beta-\alpha = \beta^2-\beta$. Thus $\beta = 2\alpha+2$. Substitution then yields

$$\alpha + 2 = (2\alpha+2)^2 - (2\alpha+2),$$

i.e., $4\alpha^2 + 5\alpha = 0$. This gives $\alpha = -5/4$ as α is not an integer.

3.

$$f(0+0^2) = f(0) + 2f(0)^2 \quad \Rightarrow \quad f(0) = 0.$$

$$f(0+1^2) = f(0) + 2f(1)^2 \quad \Rightarrow \quad f(1) = \frac{1}{2} \quad (f(1) \neq 0)$$

$$f((n+1^2) = f(n) + 2f(1)^2 \quad \Rightarrow \quad f(n+1) = f(n) + \frac{1}{2}$$

Solving this recurrence relation yields $f(n) = \frac{1}{2}n$.

4. The number of arrangements in which there are i students between Anne and Bob is $8! \times 2 \times (9-i)$: First we permute the other eight students ($8!$ ways). Then we place Anne among the first $8-i$ students ($9-i$ ways). Once this is done, there is a unique position for Bob to the right of Anne. Finally there are two ways to permute Bob and Anne.

This can also be calculated as $2\binom{8}{i}i!(9-i)!:\binom{8}{i}i!$ ways to choose and permute the i students between Bob and Anne, 2 ways to permute Bob and Anne, and $(9-i)!$ to permute the remaining students. So the probability is $\frac{9-i}{45}$ which is largest when $i=0$. So the answer is (A).

5. Suppose n is not a square. If d is a divisor of n, then so is n/d. Thus $\sigma(n)$ is even. Similarly, $\sigma(m)$ is odd if m is a square. Thus the answer is 20 as there are 10 squares.

6. We have

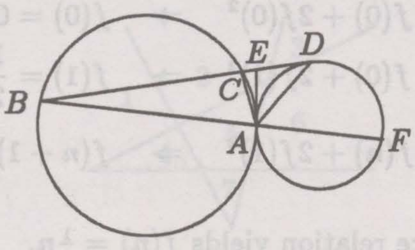
$$u_{100} = 2^{2^{4k}}$$

for some k. Noting that $36^{16} \equiv 36 \pmod{100}$, and $2^{16} = 16^4 \equiv 36 \pmod{100}$, we have

$$\begin{aligned} 2^{2^{4k}} &= \left(2^{2^4}\right)^{2^{4(k-1)}} = \left(2^{16}\right)^{2^{4(k-1)}} \equiv (36)^{2^{4(k-1)}} \pmod{100} \\ &= \left(36^{16}\right)^{2^{4(k-2)}} \equiv (36)^{2^{4(k-2)}} \pmod{100} \\ &\cdots \\ &\equiv 36 \pmod{100} \end{aligned}$$

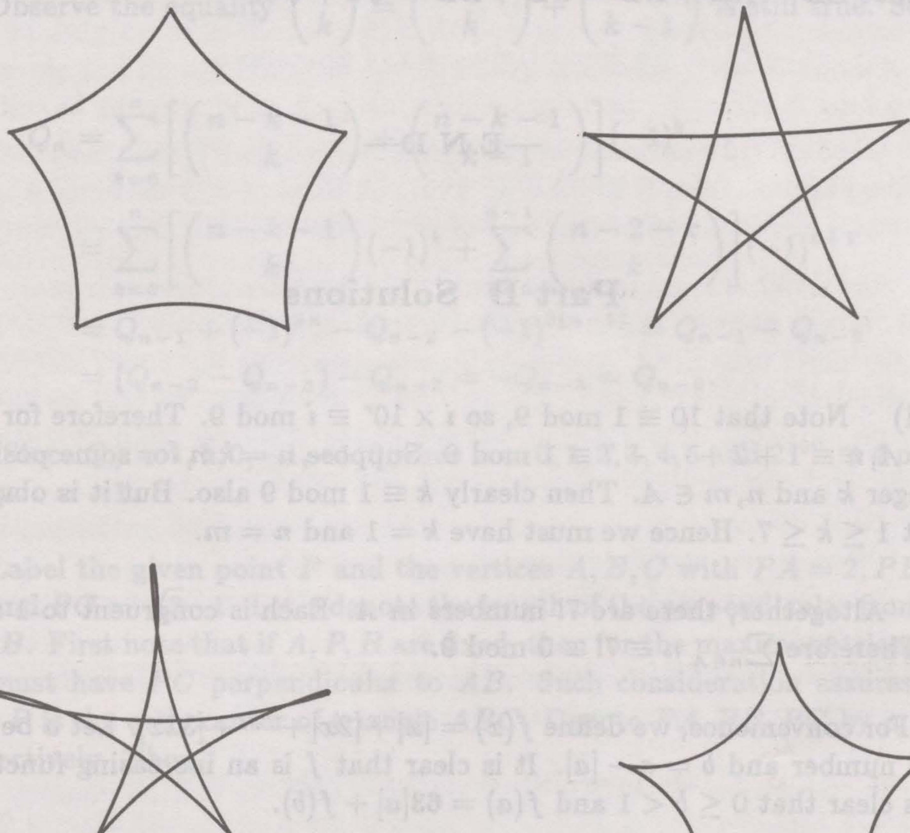
So the answer is 3. In fact the second last digit of u_n is 3 for $n \geq 4$.

7. Extend BA to meet the other circle at F and draw the common tangent EA at A (see figure below.) Let $\angle CBA = x$ and $\angle DFA = y$. Then $\angle EAD = \angle BDA = y$ and $\angle CAE = x$. Thus $\angle CAD = x + y$. Consider ΔDAB, we have $\angle DAF = x+y$. Thus $\angle CAD = 113°/2 = 56.5°$.

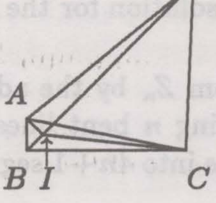


8. Let d the circumference of D. Since the pattern has 5 points whose distance from the centre is maximum, $\gcd(d, 100) = 20$, i.e., $d = 20, 40, 60$ or 80. In addition, S reaches its maximum distance from the centre first at either the second or the third point. Hence $d = 60$ or 40. But from the figure, it is obvious that the radius of the small circle is at least half that

of the large circle. So $d = 60$. The figure below shows the path traced by circles with $d = 20, 40, 60, 80$ in that order.



9. The internal bisectors of $\angle A$ and $\angle B$ meet at a point I on PB. Then $\angle AIC + \angle APC = 135° + 45° = 180°$. Thus $AICP$ is a cyclic quadrilateral. Therefore $\angle ACI = \angle APB = 2°$, whence $\angle ACB = 4°$.



10. First we note that $ABCD$ is a cyclic quadrilateral. So $\angle BCD = \angle BAD$. If E is the point on AD between A and D so that $AE = CD$, then $\Delta AEB \equiv \Delta CDB$. Since $\angle BDA = \angle BCA = 60°$, ΔBDE is equilateral, whence $AD = AE + ED = CD + BD = \sqrt{2} + \sqrt{3}$.

—END—

Part B Solutions

1 (i) Note that $10 \equiv 1 \bmod 9$, so $i \times 10^r \equiv i \bmod 9$. Therefore for any $n \in A, n \equiv 1 + 2 + \cdots + 7 \equiv 1 \bmod 9$. Suppose $n = km$ for some positive integer k and $n, m \in A$. Then clearly $k \equiv 1 \bmod 9$ also. But it is obvious that $1 \leq k \leq 7$. Hence we must have $k = 1$ and $n = m$.

(ii) Altogether, there are 7! numbers in A. Each is congruent to 1 mod 9. Therefore $\sum_{n \in A} n \equiv 7! \equiv 0 \bmod 9$.

2 For convenience, we define $f(x) = [x] + [2x] + \cdots + [32x]$. Let a be any real number and $b = a - [a]$. It is clear that f is an increasing function. It is clear that $0 \leq b < 1$ and $f(a) = 63[a] + f(b)$.

(i) Suppose $f(a) = 89$. As $f(1) = 63$ and $f(2) = 126$, we conclude $[a] = 1$ and $f(b) = 26$. Now, since $f(1/2) = 31$, $0 < b < 1/2$. In particular, this implies that $[b] = [2b] = 0$, $[4b] \leq 1, [8b] \leq 3$, $[16b] \leq 7$ and $[32b] \leq 15$. Hence $f(b) \leq 26$. Therefore $f(b) = 26$ if and only if $[32b] = 15$. This means that $47/32 \leq b < 3/2$.

(ii) When $k = 90$, we can apply similar argument as in (i) to conclude that $1 < a < 3/2$. However, as we have shown above that $f(a) \leq 63+26 = 89$, one sees that there is no solution for the equation $f(x) = 90$.

3 Z_{n+1} can be obtained from Z_n by the addition of one bent line. This bent line intersects the existing n bent lines in $4n$ points. These points divide the additional bent line into $4n+1$ segments. Each segment divides an existing region into 2. Thus $Z_{n+1} = Z_n + 4n + 1$. Hence $Z_n = 2n^2 - n + 1$ by mathematical induction.

4 Observe the equality $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ is still true. So,

$$\begin{aligned} Q_n &= \sum_{k=0}^{n}\left[\binom{n-k-1}{k} + \binom{n-k-1}{k-1}\right](-1)^k \\ &= \sum_{k=0}^{n}\left[\binom{n-k-1}{k}(-1)^k + \sum_{k=0}^{n-1}\binom{n-2-k}{k}\right](-1)^{k+1} \\ &= Q_{n-1} + (-1)^{2n} - Q_{n-2} - (-1)^{2(n-1)} = Q_{n-1} - Q_{n-2} \\ &= (Q_{n-2} - Q_{n-3}) - Q_{n-2} = -Q_{n-3} = Q_{n-6}. \end{aligned}$$

Since $Q_n = 1, 1, 0, -1, -1, 0$ when $n = 0, 1, 2, 3, 4, 5$ and $2^{100} \equiv 4 \bmod 6$, $Q_{2^{100}} = -1$.

5 Label the given point P and the vertices A, B, C with $PA = 2, PB = \sqrt{2}$ and $PC = \sqrt{3} - 1$. Let x denote the length of the perpendicular from P to AB. First note that if A, P, B are fixed, then for the maximum triangle, we must have PC perpendicular to AB. Such consideration assures us that P is the orthocenter of triangle ABC. Denote PA, PB, PC by p, q, r respectively. Then

$$\tan^2\angle ABC = (r+x)^2/(q^2 - x^2) \text{ and } \cot^2\angle BAP = (p^2 - x^2)/x^2.$$

But $\tan\angle ABC = \cot\angle BAP$, whence $(r+x)^2/(q^2-x^2) = (p^2-x^2)/x^2$. This implies $2(\sqrt{3}-1)x^3 + (10 - 2\sqrt{3})x^2 - 8 = 0$. Note that 1 is the only positive root since the polynomial function is increasing for $0 \le x$. Thus, the area is $(3+\sqrt{3})/2$.