Problems and Solutions

Problems and Solutions. The aim of this section is to encourage readers to participate in the intriguing process of problem solving in Mathematics. This section publishes problems and solutions proposed by readers and editors.

Readers are welcome to submit solutions to the following problems. Your solutions, if chosen, will be published in the next issue, bearing your full name and address. A publishable solution must be correct and complete, and presented in a well-organised manner. Moreover, elegant, clear and concise solutions are preferred. Solutions should reach the editors before 30 October, 1991.

Readers are also invited to propose problems for future issues. Problems should be submitted with solutions, if any. Relevant references should be stated. Indicate with an ° if the problem is original and with an * if its solution is not available.

All problems and solutions should be typewritten double-spaced, and two copies should be sent to the Editor, Mathematical Medley, c/o Department of Mathematics, The National University of Singapore, 10 Kent Ridge Crescent, Singapore 0511.

Problems

P19.1.1. *Proposed by Leong Chong Ming, Nanyang Junior College.*

For $n \geq 1$, let $s_n = \frac{1}{a_1, a_2 \ldots a_m}$ where the sum is taken over all m and all finite sequences of the positive integers $a_1, a_2, \ldots, a_m$ such that $a_1 = n$ and $a_{i+1} \leq a_i - 2$ for $1 \leq i \leq m-1$. For example,

$$s_6 = \frac{1}{6} + \frac{1}{6 \cdot 4} + \frac{1}{6 \cdot 3} + \frac{1}{6 \cdot 2} + \frac{1}{6 \cdot 1} + \frac{1}{6 \cdot 4 \cdot 2} + \frac{1}{6 \cdot 4 \cdot 1} + \frac{1}{6 \cdot 3 \cdot 1 \cdot}$$

Find $\lim\limits_{n \to \infty} s_n$.

P19.1.2 *Proposed by Lee Peng Yee, National University of Singapore.*

Without using calculus, prove that $(v - u)\sin u \leq \cos u - \cos v$ for $0 \leq u \leq v \leq \frac{\pi}{2}$.

P19.1.3 *Proposed by Choi Kwok Pui, National University of Singapore.*

Prove that if $a_k > 0$, then

$$\sum_{k=1}^{\infty} \frac{a_k}{(1+s_k)^2} \text{ converges}$$

where $s_k = \sum_{i=1}^{k} a_i$.

The following problems are questions from the International Mathematical Olympiad held in Beijing, China, July 1990

P19.1.4

Two chords AB, CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The line through E tangent to the circumcircle through D, E, M intersects the lines BC, AC at F, G respectively. If $\frac{AM}{AB} = t$, find $\frac{EG}{EF}$ in terms of t.

P19.1.5

Let $n \geq 3$ and consider a set E of $2n - 1$ distinct points on a circle. Suppose that exactly k of these points are to be coloured black. Such a colouring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such colouring of k points of E is good.

P19.1.6

Determine all integers $n > 1$ such that $\frac{2^n+1}{n^2}$ is an integer.

P19.1.7

Let Q^+ be the set of positive rational numbers. Construct a function $f : Q^+ \to Q^+$ such that $f(xf(y)) = \frac{f(x)}{y}$ for all x, y in Q^+.

P19.1.8

This is a 2-person game.
An initial integer $n_0 > 1$ is given, and two players A and B choose integers $n_1, n_2, n_3, \ldots$ alternatively according to the following rules:

Knowing n_{2k}, A chooses any integer n_{2k+1} such that $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$. Knowing n_{2k+1}, B chooses any integer n_{2k+2} such that $\frac{n_{2k+1}}{n_{2k+2}}$ is a positive power of a prime.

Player A wins by choosing the number 1990, player B wins by choosing the number 1. For what values of n_0 does

(a) A win ?

(b) B win ?

(c) nobody win ?

P19.1.9

Prove that there exists a convex 1990 - gon with sides of length

$$1^2, 2^2, 3^2, \ldots, 1989^2, 1990^2, \text{ in some order,}$$

and whose interior angles are equal.

Solutions

P18.1.1 *Solution by the Proposer*

First we let

$$f(x) = \frac{\left(\frac{A+x^p}{n}\right)^{1/p}}{\left(\frac{B+x^q}{n}\right)^{1/q}}.$$

By differentiation, the minimum is attained by

$$x = \left(\frac{A}{B}\right)^{1/(p-q)}.$$

If it is given that

$$\left(\frac{A}{n-1}\right)^{1/p} \geq \left(\frac{B}{n-1}\right)^{1/q},$$

we have

$$f(x) \geq n^{p-q} \frac{A^q}{(B^{p/(p-q)} + A^{q/(p-q)})^{(p-q)}} \geq 1.$$

Now, by letting $A = a_1^p + a_2^p + \ldots + a_{n-1}^p$, $B = a_1^q + a_2^q + \ldots + a_{n-1}^q$, and by mathematical induction, the proof is complete.

P18.1.6 ***Solution by Ng Weng Leong, Raffles Junior College.***

There are two cases to consider, n odd and n even. Suppose $n = 2k + 1$, $k \geq 3$, $k \in \mathbb{Z}_+$, then a convex hexagon may be constructed as shown in Figure 1.

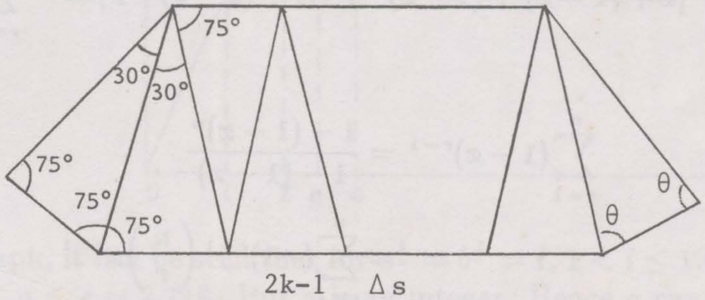


Figure 1

If $n = 2k$, $k \geq 3$, $k \in \mathbb{Z}_+$, then the hexagon is shown in Figure 2.

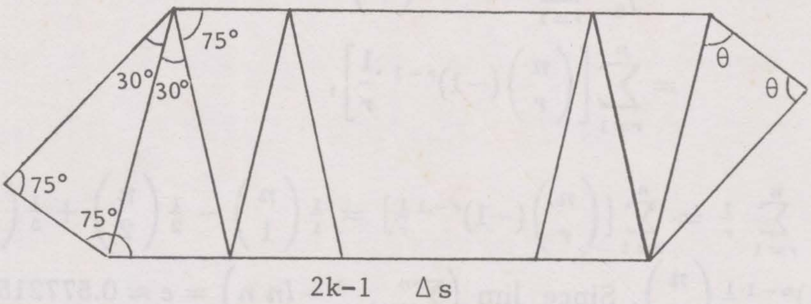


Figure 2

Each of the above constructions use congruent isosceles Δs with angles $30°$, $75°$, $75°$. The choice of $\theta = 75°$ is quite arbitrary. The construction will work so long as $60° < \theta < 90°$.

P18.2.2 ***Solution by Leong Chong Ming, Nanyang Junior College. Also solved by Gan Wee Teck, student, Hwa Chong Junior College (1990).***

Firstly, we consider

$$\int^1 (1 + u + u^2 + \ldots + u^{n-1})du = \sum_{r=1}^{n} \frac{1}{r} \qquad (*)$$

Substituting $u = 1 - x$ into (*) we get

$$\int_0^1 [1 + (1-x) + (1-x)^2 + \ldots + (1-x)^{n-1}]dx = \sum_{r=1}^{n} \frac{1}{r}.$$

Now

$$\sum_{r=1}^{n}(1-x)^{r-1} = \frac{1-(1-x)^n}{1-(1-x)}$$

$$= \sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}x^{r-1}$$

Hence

$$\int_0^1 [1 + (1-x) + (1-x)^2 + \ldots + (1-x)^{n-1}]dx$$

$$= \int_0^1 \sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}x^{r-1}dx$$

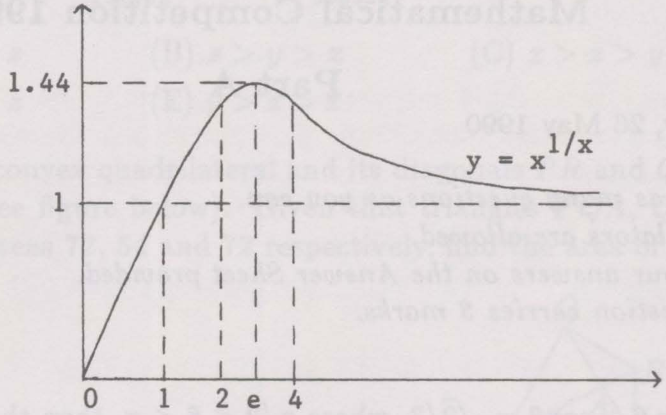
$$= \sum_{r=1}^{n}\left[\binom{n}{r}(-1)^{r-1}\frac{1}{r}\right],$$

i.e. $\sum_{r=1}^{n} \frac{1}{r} = \sum_{r=1}^{n}[\binom{n}{r}(-1)^{r-1}\frac{1}{r}] = \frac{1}{1}\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \ldots + (-1)^{n-1}\frac{1}{n}\binom{n}{n}$. Since $\lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{r} - ln\ n\right) = c \approx 0.5772156$, the Euler constant

$$\sum_{r=1}^{n}\frac{1}{r} \approx ln\ n \quad \text{for large } n.$$

P18.2.3 *Solution by Ng Weng Leong, Raffles Junior College. Also solved by Chen Junxiang, Anglo-Chinese School.*

$a^b = b^a \Leftrightarrow a^{\frac{1}{a}} = b^{\frac{1}{b}}$. Consider the function $y = x^{\frac{1}{x}}$



From the graph, it can be seen that for $a^{\frac{1}{a}} = b^{\frac{1}{b}} = i, 1 < i \leq 1.44$. Taking $b > a \Rightarrow 1 < a \leq e \approx 2.718$. But a is an integer. Hence a can only be 2. Therefore there is no other solution.

Singapore Mathematical Society

Inter-Secondary School
Mathematical Competition 1990

Part A

Saturday, 26 May 1990 1000–1200

Attempt as many questions as you can.
No calculators are allowed.
Circle your answers on the Answer Sheet provided.
Each question carries 3 marks.

1. If $\sin\theta + \cos\theta = \sqrt{2}/3$, where $\pi/2 < \theta < \pi$, then the value of $\sin\theta - \cos\theta$ is

(A) $-\frac{4}{3}$ (B) $\frac{4}{3}$ (C) $-\frac{3}{4}$ (D) $\frac{3}{4}$ (E) None of the preceding

2. If $x^2 + kx + 1 = 0$ and $x^2 - x - k = 0$ have exactly one common real root, then the value of k is

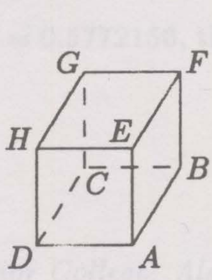
(A) -1 (B) 3 (C) 1 (D) 2 (E) None of the preceding

3. If $\log_8 a + \log_4(b^2) = 5$ and $\log_8 b + \log_4(a^2) = 7$, then $\log_2(ab)$ is

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

4. In a unit cube $ABCDEFGH$ as shown, find the perpendicular distance from E to the plane AFH.

(A) $\sqrt{3}$
(B) $\sqrt{2}$
(C) $\frac{\sqrt{2}}{2}$
(D) $\frac{\sqrt{3}}{3}$
(E) None of the preceding



5. Find the exact value of

$$\sin^4 22.5^\circ + \sin^4 67.5^\circ + \sin^4 112.5^\circ + \sin^4 157.5^\circ .$$

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{1}{2}$ (D) $\frac{2}{\sqrt{2}}$ (E) None of the preceding

6. If $x = \frac{111110}{111111}$, $y = \frac{222221}{222223}$ and $z = \frac{333331}{333334}$, then

(A) $x > y > z$ (B) $z > y > x$ (C) $x > z > y$

(D) $y > x > z$ (E) $y > z > x$

7. $PQRS$ is a convex quadrilateral and its diagonals PR and QS intersect at A (see figure below). Given that triangles PQA, QRA and RSA have areas 72, 54 and 72 respectively, find the area of ΔSPA.

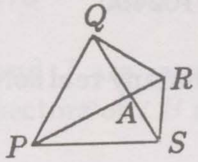
(A) 54

(B) 74

(C) 96

(D) 108

(E) None of the preceding



8. A six-digit number $XYXYXY$ is equal to 5 times the product of three consecutive odd numbers. Find the sum of these three odd numbers.

(A) 123 (B) 117 (C) 115 (D) 111 (E) 101

9. Find the value of

$$\frac{1}{1\cdot\sqrt{2}+2\cdot\sqrt{1}} + \frac{1}{2\cdot\sqrt{3}+3\cdot\sqrt{2}} + \frac{1}{3\cdot\sqrt{4}+4\cdot\sqrt{3}} + \cdots + \frac{1}{99\cdot\sqrt{100}+100\cdot\sqrt{99}}.$$

(A) $\frac{1}{10}$ (B) $\frac{1}{\sqrt{99}}$ (C) $\frac{11}{10}$ (D) $\frac{9}{10}$ (E) $\frac{10}{\sqrt{99}}$

10. Which of the following numbers is prime?

(A) $17231^{60}+1$ (B) $1\underbrace{00\cdots00}_{2000}1$ (C) $2^{2^4}+1$

(D) $2^{154}-1$ (E) $2^{154}+1$

11. In how many ways can 5 different prizes be awarded to 4 students so that each student has at least one prize?

(A) 480 (B) 3600 (C) 1800 (D) 240 (E) 120

12. Among all the nine-digit numbers formed from the digits 1, 2, ..., 9 without repetition, what proportion is divisible by 18?

(A) $\frac{1}{18}$ (B) $\frac{4}{9}$ (C) $\frac{1}{9}$ (D) $\frac{5}{9}$ (E) $\frac{17}{18}$

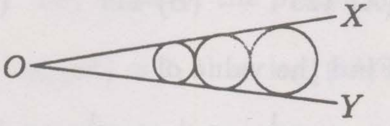
13. For any non-empty set of numbers A, let S_A be the sum of all the numbers in A. Find the sum of all the S_A's as A runs through all the non-empty subsets of $\{1,2,\ldots,10\}$.

(A) 14080 (B) 28160 (C) 56320
(D) 102400 (E) None of the preceding

14. How many real solutions are there of the equation $2^{4x}+(2^x-2)^4-34=0$?

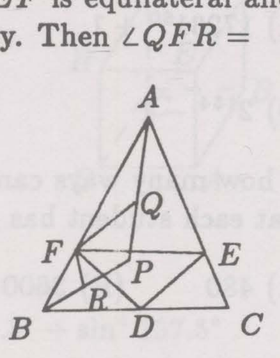
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

15. The figure below shows three circles touching one another successively with OX, OY their common tangents. If the radii of the circles are r_1, r_2, r_3, where $r_1 < r_2 < r_3$, then



(A) $r_2^2 = 2r_1r_3$
(B) $r_2 = \frac{1}{2}(r_1 + r_2)$
(C) $r_2 = \frac{1}{3}(2r_3 + r_1)$
(D) $r_2^2 = r_1r_3$
(E) None of the preceding

16. In ΔABC, $\angle A = 25.5°$ and $\angle B = 74.5°$. Let D, E, F be the feet of the perpendiculars from a point P inside the triangle to the sides BC, AC, AB respectively. Suppose ΔDEF is equilateral and Q, R are the mid-points of AP, BP respectively. Then $\angle QFR =$



(A) 100°
(B) 134.5°
(C) 140°
(D) 145.5°
(E) 160°

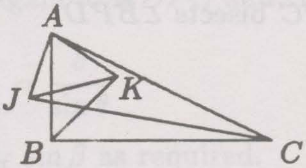
17. Let q, r, b be three consecutive integers with $b \geq 2$ and $q < r < b$. In the numeral system with base b, the square of $rrrr$ is

(A) $rrrq0001$ (B) $rrq00001$ (C) $rrrrq001$
(D) $rrrqq001$ (E) $rrq0001$

18. How many numbers n between 1 and 1000 are divisible by $[\sqrt[3]{n}]$? ($[x]$ denotes the largest integer $\leq x$. For example, $[3] = 3$, $[3.2] = 3$, $[\sqrt[3]{10}] = 2$.)

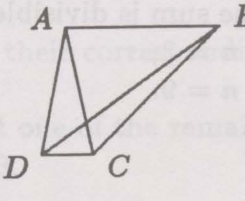
(A) 170 (B) 172 (C) 174 (D) 176 (E) 178

19. In ΔABC, $\angle B = 90°$, $\angle C = 2°$. Suppose K and J are respectively the feet of the perpendiculars from A to the bisectors of $\angle B$ and $\angle C$. Then $\angle KJC =$



(A) 1°
(B) 2°
(C) 3°
(D) 4°
(E) None of the preceding

20. In quadrilateral $ABCD$, $AB \parallel DC$, $AB = AC = AD = \sqrt{3}$, $BC = \sqrt{2}$. Find BD.



(A) $\sqrt{2} + \sqrt{3}$
(B) $\sqrt{6}$
(C) $3\sqrt{2}$
(D) $2\sqrt{3}$
(E) None of the preceding

—E N D—

Answers to Part A: 1B, 2D, 3E, 4D, 5A, 6E, 7C, 8D, 9D, 10C, 11D, 12B, 13B, 14B, 15D, 16C, 17A, 18B, 19A, 20E.

Singapore Mathematical Society

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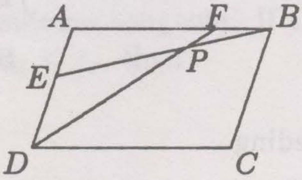
Mathematical Competition 1990

Part B

Saturday, 26 May 1990 1200–1300

Attempt as many questions as you can.
No calculators are allowed.
Each question carries 20 marks.

1. In the figure below $ABCD$ is a parallelogram and E, F are points on the sides AD, AB respectively. Suppose the line segments BE and DF are equal in length and they meet at the point P. Prove that PC bisects $\angle BPD$.



2. Given any $2n-1$ positive integers, prove that there are n of them whose sum is divisible by n for

(i) $n = 3$;
(ii) $n = 9$.

—E N D—

Part B Solutions

1. Let X and Y be the feet of the perpendiculars from C to DF and BE respectively. Then CP bisects $\angle BPD$ if and only if $CX = CY$. Denoting the area of a polygon $ABC\ldots$ by $(ABC\ldots)$, we have $(DFC) = (BCE) = \frac{1}{2}(ABCD)$. Since $DF = BE$ and CX and CY are the altitudes of ΔDFC and ΔBCE, we have $CX = CY$ and the proof is complete.

Second solution: Let $AB = a$, $AD = b$, $\angle AFD = x$, $\angle AEB = y$. Then by applying the sine rule to ΔAEB and ΔAFD, we have

$$\frac{a}{\sin y} = \frac{b}{\sin x}. \qquad (1)$$

Since alternate angles are equal, $\angle PDC = x$ and $\angle PBC = y$. Let $\angle BPC = \theta$ and $\angle DPC = \beta$. Apply sine rule again to ΔDPC and ΔBPC, we have

$$\frac{PC}{\sin x} = \frac{a}{\sin\beta} \quad \text{and} \quad \frac{PC}{\sin y} = \frac{b}{\sin\theta}. \qquad (2)$$

From (1) and (2), it is easily seen that $\sin\theta = \sin\beta$ as required.

2 (i) Let $a_1, a_2, \ldots, a_5$ be any 5 positive integers and $r_1, r_2, \ldots, r_5$ be their respective remainders when divided by 3. Without loss of generality, we can assume that

$$0 \le r_1 \le r_2 \le \cdots \le r_5 \le 2.$$

If three of the r_i's are equal, then the sum of their corresponding a_i's is divisible by 3.

If not, then $r_1 = 0$ and $r_5 = 2$ and at least one of the remaining r_i's, say r_2, is 1. Then $a_1 + a_2 + a_5$ is divisible by 3.

(ii) Let $b_1, b_2, \ldots, b_{17}$ be the given positive integers. Then by (i), we can find three integers whose sum is divisible by 3, say

$$b_1 + b_2 + b_3 = 3c_1.$$

From the remaining 14 numbers, we can find three integers whose sum is divisible by 3, say

$$b_4 + b_5 + b_6 = 3c_2.$$

Repeating this process, we have, without loss of generality

$$b_7 + b_8 + b_9 = 3c_3,$$
$$b_{10} + b_{11} + b_{12} = 3c_4,$$
$$b_{13} + b_{14} + b_{15} = 3c_5.$$

From the 5 positive integers $c_1, \ldots, c_5$, we can find three say, c_1, c_2, c_3, whose sum is divisible by 3. Thus $b_1 + b_2 + \cdots + b_9$ is divisible by 9.

—E N D—