A New Science : Chaos

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In the new movie Jurassic Park [C], Malcolm, a mathematician specialized in Chaos Theory, explained that Hammond's project "is another apparently simple system - animals within a zoo environment - that will eventually show unpredictable behaviour". Then he proclaimed, "There is a problem with that island. It is an accident waiting to happen." [C] Soon the accident happened during a dark and stormy night.

Let us neglect at the ambiguity on what exactly the nonlinear system is and why it is chaotic. It is just amazing that Chaos as a new science has been understood and accepted by Hollywood so quickly and so easily. I am sure that the writer was inspired by Chaos Theory to write such a story. Chaos Theory has found its application here. We will introduce more examples next for you to understand Chaos. You can find more examples in [GL]. For an introduction to Chaos Theory, you can read [GU].

What is Chaos?

A dynamical system is a deterministic system whose future is determined by its past. Differential equations are examples of dynamical systems. If a differential equation is linear, we can solve the equation completely. If it is nonlinear, normally we are not able to find a general solution. Then can we find a special solution if the initial value is given? Yes. We can use computers to solve the equation numerically. If the initial value has 10 decimal places, e.g. 1.6101235791, can we just take 1.6101 or 1.6 for convenience? By common sense, it should be all right. But Lorenz found that the difference can be quite big. The reason is that nonlinear dynamical systems can be chaotic.

In 1960, Lorenz created a toy weather model with 12 differential equations. He then tried to solve these equations on a machine, a Royal McBee, in order to understand the weather. At the time, computers were still in the early stage. They could not run as fast as the modern computers do. Besides the calculation could be interrupted if one of the vacuum tubes went bad. When that happens, one can either run the program from the beginning or continue the computation by picking up a number from the print out as a new initial value. One will not expect any difference in the final result.

But when this happened to Lorenz, a miracle happened. He found that the new results are quite different from the previous ones. What has happened? Is there any vacuum tube broken? He checked it out and found no broken vacuum tubes. When he examined closely, he found that the numbers on the print out are of three decimal places, .506, while the corresponding numbers in the memory are of six decimal places, .506127.

The difference of the two numbers is one of a thousand. How can it cause so much difference in the final result? He didn't expect to get gold when he mixed sodium and chlorine. But he did get gold : Chaos. It was the hidden characteristics of the system that made the difference.

When we apply the phenomenon to the weather, it implies that the long term forecasting is impossible since a small change of nature may cause big changes of the weather. Such a property has been named as "Butterfly Effect", i.e. the fluttering of a butterfly in New York brings a snowfall in Beijing in the next year. The technical name for the Butterfly Effect was called sensitive dependence on initial conditions. It can be put in another way: the weather system is sensitively dependent on the initial conditions.

Lorenz put the weather aside and looked for even simpler ways to produce this complex behaviour. He found one in three dimensional space that is represented by the following equations:

$$\begin{cases} \dfrac{dx}{dt} = s(-x+y), \\ \dfrac{dy}{dt} = rx - y - xz, \\ \dfrac{dz}{dt} = -bz + xy. \end{cases}$$

This is the simplest nonlinear system one can find in three dimensional phase space. But it is sensitively dependent on the initial conditions.

Chaos has two essential characteristics. First, the system is sensitively dependent on the initial conditions. Second, the complex behaviour of the system has an underlying order. Lorenz has drawn the itineraries of (x, y, z) generated from the above equations on the phase space. He found that the itineraries were attracted to an attractor which does not have a regular figure (see figure 1). Such a figure is called a Lorenz attractor.

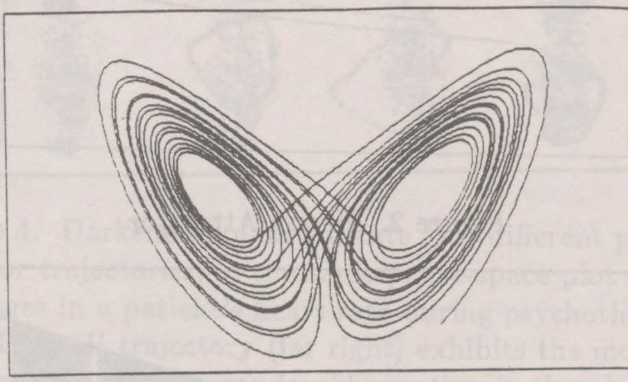

Figure 1. Lorentz Attractor

An attractor is a set that is invariant by the action of the system and such that the itineraries of the points in its neighborhood converge to it. An attractor can be a point, a circle or an irregular figure. The shape of the irregular figure appeared strange to the early founders. It was thus named a strange attractor. Even though the shape looks strange, the attractor still has good properties. There are laws to obey inside the strange attractors. This explains the second characteristic of chaos.

The Lorenz equations are of three variables. The phase space is three dimensional. Lorenz equations are the simplest nonlinear equations one can find. This system is chaotic. Many more nonlinear systems have been found to be chaotic.

Henon found a system of two variables that has a strange attractor as follows. It is a simple nonlinear system of two variables that is chaotic. The strange attractor of this system is called Henon attractor (see figure 2).

$$\begin{cases} x_2 = 1 + y_1 - a x_1^2 \\ y_2 = b x_1 \end{cases}$$

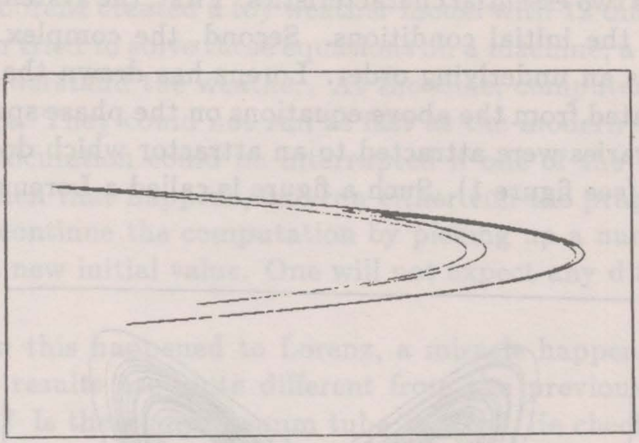

Figure 2. Henon Attractor

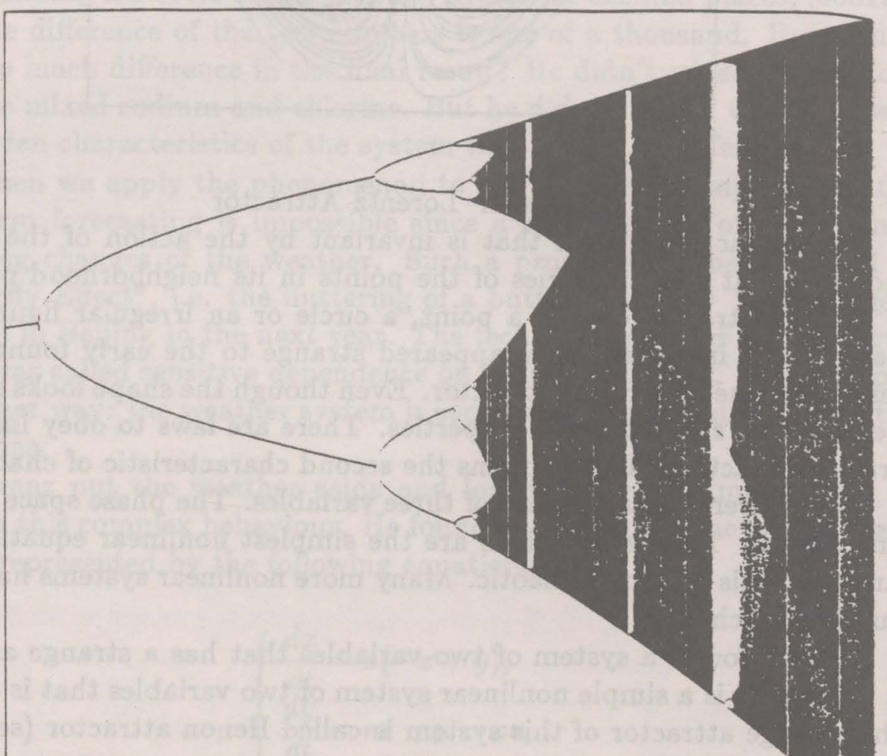

Figure 3. Bifurcation of $rx(1-x)$

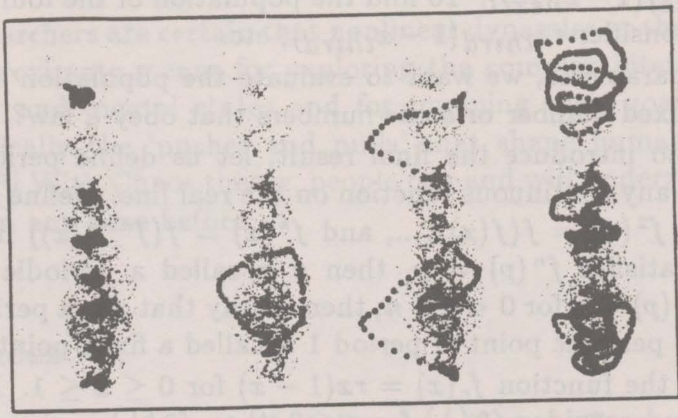

Figure 4. Darkened points indicate four different patterns, or trajectories, in the same phase-space plot of changes in a patient's heart rate during psychotherapy. Type IV trajectory (far right) exhibits the most complex, spontaneous, randomlike motion in the phase space and may reflect specific kinds of mental states patients experience during therapy, say researchers.

The Application of Chaos Theory

Ever since the discovery of the existence of chaos in simple nonlinear systems, mathematicians and scientists have found more and more nonlinear systems that are chaotic. Surprising messages are going out : chaos is everywhere. Mathematicians and physicists are studying chaos. Biologists and economists are also doing research on chaos. Even linguists and doctors are working on chaos. The following are two examples.

Let us first look at the simplest ecological question of how single populations behave over time. Let x denote the population of some kind of fish at a certain year. Let x_{next} denote the population in the next year. Then ecologists have established a model with the following expression

$$x_{next} = rx(1-x)$$

where r is the rate of growth which corresponds to the amount of food, the amount of heating or the amount of friction, etc. In order to find the population of the third year, we need only consider $x_{third} = rx_{next}(1 - x_{next})$. To find the population of the fourth year we need only to consider $rx_{third}(1 - x_{third})$, etc.

Given a parameter, we want to evaluate the population after many years. Is it a fixed number or some numbers that obey a law?

In order to introduce the final result, let us define periodic orbits first. Let f be any continuous function on the real line. Define $f^0(x) = x$, $f^1(x) = f(x)$, $f^2(x) = f(f(x)), \ldots$, and $f^k(x) = f(f^{k-1}(x))$ inductively. If a point p satisfies $f^n(p) = p$, then p is called a periodic point. If furthermore $f^i(p) \neq p$ for $0 < i < n$, then we say that p is a periodic point of period n. A periodic point of period 1 is called a fixed point.

Let f_r be the function $f_r(x) = rx(1 - x)$ for $0 \leq x \leq 1$. If we start with $x_0 = \frac{1}{2}$ and consider $f_r^n\left(\frac{1}{2}\right)$ for $r \leq 3$, then $f_r^n\left(\frac{1}{2}\right)$ is close to a fixed point when n is very large, i.e., the limit of $f_r^n\left(\frac{1}{2}\right)$ is a fixed point. When $3 < r < 3.5$, $f_r^n\left(\frac{1}{2}\right)$ approaches a periodic orbit of period 2. We say that it bifurs at 3. When r gets bigger, we find that the limit of $f_r^n\left(\frac{1}{2}\right)$ continues to bifur. Thus the periods are $2, 4, 8, 16, \ldots$, etc until r reaches a number after which it turns out to be chaotic. If we were finding an animal population governed by this simplest nonlinear system, you would think that the changes from year to year were absolutely random.

This model was one of the few chaotic models understood by mathematicians. It has been understood completely. Scientists are now doing research on real life models which are not mathematically proved to be chaotic. But by using the computer, one can see from the attractors that they appear to be chaotic. The following is an example [P].

Two scientists, Dana J. Redington and Steven P. Reidbord recorded the heartbeat of a woman who was seeking counselling and embarked on a series of weekly psychotherapy sessions to quell the unusually prolonged aftershock of her loss of her husband. "During the 50-minute session, the conversation wandered; so did the woman's heart rate" [P]. They plotted the second by second changes of heart rate in three dimensional phase space to highlight the amount of variability.

A truly random heart rate will fill the phase space with an amorphous cloud of data points in no particular order. Conversely a completely stable heart rate - that of a person under deep anaesthesia, for example - would fill only a small, point-like region of the phase space.

But in this case, the data looks like something in between these extremes. One can observe four different patterns at different time intervals. The last one shown in the figure is most complex and is considered chaotic.

The trajectories tend to emerge when the patient seems more defensive or anxious.

"Researchers are certain that nonlinear dynamics or the Chaos theory is the approximate means for exploring the complex interconnections of physiology and mental states and for pursuing their goal of describing mathematically the 'pushes and pulls' that shape human thought and action." [P] With Chaos theory, people can and will understand the world better than any time before.

References

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[P] D. Pendick, *Chaos of the Mind*, Science-News, 143(1993), Pages 129-144.