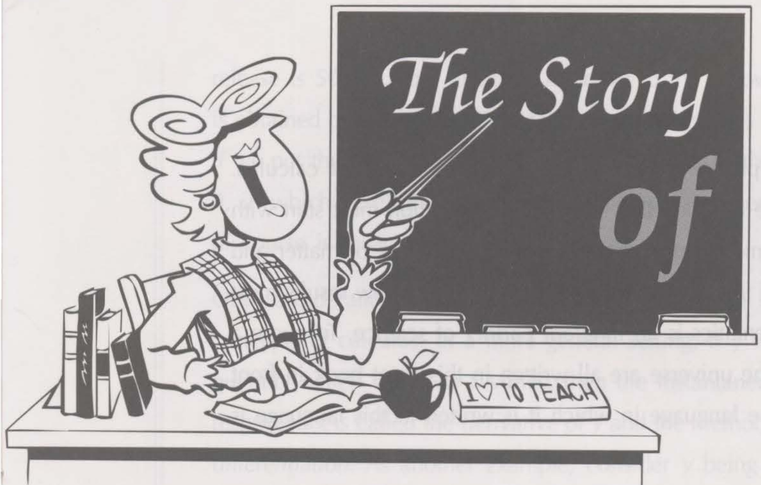

The Story of Calculus II[1]

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In the last episode, we mentioned the work of the father and son mathematician team Zu Chongzhi and Zu Geng during the North-South Period (AD 420—589) and how one of their results (now known as Cavalieri's Principle) was rediscovered in Europe about one thousand years later and was used by Kepler to calculate the area of an ellipse. So why was Kepler interested in the area of an ellipse?

4. The Rise of Mathematics in Western Europe

Two millenia prior to Kepler, the ancient Greeks had discovered ellipses. However, their interest in them was purely as an exercise in geometry; it was Kepler who brought the ellipse into the realm of practical science. Curiosity in the mysteries of the universe led Kepler to Tycho Brahe, the leading astronomer of his time, to be his assistant. Upon the sudden demise of Tycho Brahe two years later, Kepler inherited his position, and with it the richest and most elaborate depository of astronomical data in Europe at the time, and Kepler was determined to find the laws that governed the universe from this wealth of data. Kepler kept to the highest standards of empiricism, rejecting one theory after another only for minute discrepancies with the data. Finally in the year 1609, he published two celebrated Laws:

(I) The planets follow elliptic orbits around the sun, with the sun as a focus;

(II) the straight line segment joining the sun and the planet sweeps out equal areas in equal time intervals.

Ten years later, he published the Third Law:

(III) The square of the period of the planet's revolution around the sun is proportional to the cube of the semi-major axis of the elliptic orbit.

While the Polish astronomer Copernicus had made the ground-breaking proposal in 1543 that planets revolved around the sun, he was unable to break completely from ancient Greek influences and believed that they did so in circular orbits with constant speed. Kepler's discoveries not only turned the world to the investigation of non-circular and non-uniform motions, they contributed directly to the development of calculus. It was in order to understand Kepler's Three Laws better that led Newton to his theory of mechanics, for which calculus was a prerequisite.

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Contemporarily, the Italian physicist Galileo Galilei also played a key role in the development of calculus. Basically, he initiated the "mathematization" of science. He believed that scientific investigation must start with natural phenomena. From the intricacies of natural phenomena one extracts the basic concepts of matter and motion. By using mathematics, new results can be deduced from these basic concepts, and the new results must then be verified by laboratory experimentation. Thus mathematics is an inherent partner of science. To quote a famous saying Galileo made in 1610: "The mysteries of the universe are all written in this great book in front of us, and in order to understand it, we must first learn the language in which it is written ... this language is mathematics."

As a result of the Renaissance of the 15th century, Europe in the 17th century was imbued with the keen desire to understand nature. This desire stimulated the thinking and widened the horizon of people. At the same time, advances in industry and commerce brought new and pressing scientific and technological problems. Specifically, these problems involved variable quantities which could not be tackled with the "static" mathematics of the time. Naturally, these "dynamical" challenges attracted most of the first-rate minds of the day.

5. Differentiation and Integration

Let us first consider an important concept, namely, rate of change. For simplicity of exposition, let us take the example of the rate of change of distance versus time – in other words, speed. First, we must distinguish between two types of speed: average speed and instantaneous speed. Average speed is a simple concept: it is the quotient of the distance covered in a certain time interval divided by the length of that time interval. But what is often of more interest is instantaneous speed. For example, when a bullet hits a target, the average speed of the bullet does not tell much, it is the instantaneous speed at impact which carries the force! How can we understand instantaneous speed? Suppose a bullet is moving forward, and between the time instant T and $T + 0.5$ sec it has travelled 207.5 m. Then the average speed for this time period would be 415 m/s. If we shorten the time of observation to between T and $T + 0.1$, and suppose the bullet has travelled 41.9 m in this time interval, then the average speed would be 419 m/s. Repeating this now for the time interval between T and $T + 0.01$, we may observe that the distance covered becomes 4.199 m, giving an average speed of 419.9 m/s. Keep on shortening the time interval this way, the average speed will tend towards a certain number. This number should be the instantaneous speed of the bullet at time T. You may argue: "If an instant is a time interval of length zero, how can the bullet change position in an instant and how can there be any (instantaneous) speed?" Indeed, this is an archaic argument first put forward by the ancient Greeks in the 5th century BC. Let us not get bogged down by its debate – anyone who has run into an object and got bruised would not doubt the evidence of instantaneous speed.

The following principle was observed by Galileo: When an object falls freely from rest, the distance fallen is directly proportional to the square of time. When a quantity varies with another quantity, this relationship is called a function in mathematics. Let's denote by S the distance fallen, and by T the time in which the fall takes place, then we say that S is a function of T. If we measure S in metres and T in seconds, then Galileo's principle can be expressed as $S = 5T^2$ (5 is an approximate value). We may ask what is the object's instantaneous speed after 5 seconds. (Incidentally, the average speed for the first 5 seconds is 25 m/s as the object falls 125 m.) The instantaneous speed can be worked out as follows: We observe that when $T = 5$, $S = 125$; and when

$$T = 5 + h,\ S = 5(5 + h)^2 = 125 + 50h + 5h^2. \qquad (1)$$

Thus the average speed in this time interval of length h is $50 + 5h$ m/s. As h gets smaller and smaller, the average speed tends to 50 m/s. Thus, the instantaneous speed of the free-falling object five seconds after its release is 50 m/s. Your attention is drawn to the following very important fact: It seems that the instantaneous speed is obtained by putting $h = 0$, in (1); however, this does not concur at all with the reasoning we have just given. Indeed, if we put the length of the time interval $h = 0$, then the difference S would also be 0, in which case when we calculate the average speed we would get 0/0, which of course is meaningless.

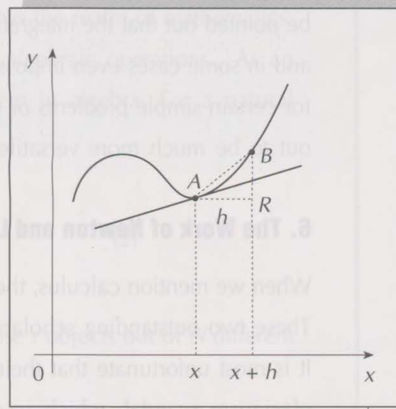

Diagram 1

Our consideration of the instantaneous speed above in fact contains the essential concepts of calculus. In a more general setting, if y is a function of x, i.e., y is a quantity which varies as x varies, then the instantaneous rate of change of y with respect to x is called the *derivative* of y and the method for its computation is called *differentiation*. As another example, consider y being atmospheric pressure and x being height, then the derivative of y with respect to x measures the instantaneous rate of change of atmospheric pressure with respect to height (note that here the word "instantaneous" does not refer to a time instant but a height level). The usual mathematical notation for this derivative is $\frac{dy}{dx}$, and geometrically it can be described as follows: Consider the graph of the function y with respect to x in the usual x - y coordinate system. Note that the average rate of change between x and $x + h$ is the slope of the secant joining the points A and B as indicated in Diagram 1.

As we shorten h repeatedly, the secant approaches the tangent R to the graph at x (see Diagram 1). Thus, the instantaneous rate of change is the slope of the tangent.

Computing the slope of a tangent was an important area of research in the 17th century. Around 1630, the French mathematician Fermat proposed a systematic method for their computation, which was basically the method we described in the last paragraph. In this way, Fermat computed the slopes of the tangents to the curves $y = x^2$, $y = x^3$, $y = x^4$, etc., and found that at the point $x = a$, they were $2a$, $3a^2$, $4a^3$, etc., respectively. At the same time, Fermat made important contributions to the computation of the area under a curve. He considered a lot of very narrow rectangles under the curve (see Diagram 2). The sum of their areas would then give an estimate of the area under the curve from below. As these rectangles became narrower and narrower (and thus more and more of them would be needed), the sum of their areas would tend to be the actual area under the curve. Using this idea, Fermat computed the areas under the graphs $y = x^2$, $y = x^3$, $y = x^4$, etc., and found that between $x = 0$ and $x = a$, they were $a^3/3$, $a^4/4$, $a^5/5$, etc., respectively. This is in fact the fundamental concept of integral calculus. At this point, the groundwork for differential and integral calculus had been laid.

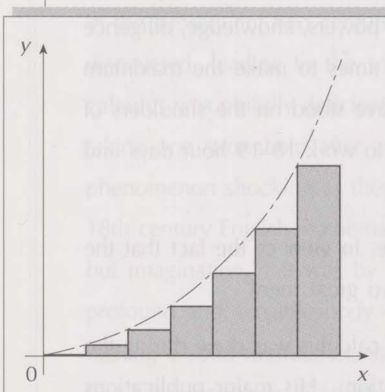

Diagram 2

Judging from Fermat's calculations, it is hard to believe that he was not already aware of the relationship between differential and integral calculus. However, in the mathematical literature, the first to note the connection seems to be the English mathematician Barrow, who was Newton's teacher at Cambridge University. This was later clarified by Newton and the German mathematician Leibniz, and is known today as the *Fundamental Theorem of Calculus*. Very roughly, this theorem says that if we first integrate a function and then differentiate the integral, we recover the original function. To be more precise, let f be a continuous function on the closed interval $[a, b]$, and let

$$F(x) = \int_a^x f(t)\,dt,$$

then f is the derivative of F. In fact, today, most people compute an integral by using this theorem – first, find the indefinite integral (also called the primitive, which is a function whose derivative is the given one) for which there is a host of methods, then evaluate it at the upper and lower limits and take the difference. For example, for $f(x) = x$, an indefinite integral is $x^2/2$, and thus

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$$

However, we must bear in mind that this is a *technique* for computing an (definite) integral and is not its *definition*. An (definite) integral is, by definition, the limit of an infinite sum (see explanations given in the last paragraph). It must be pointed out that the integration technique using the Fundamental Theorem of Calculus is in practice rarely effective, and in some cases even impossible as was demonstrated by mathematicians in the first half of the 19th century, except for certain simple problems or textbook exercises! By formalizing these concepts, differential and integral calculus turn out to be much more versatile than just computing speed or area.

6. The Work of Newton and Leibniz

When we mention calculus, the names of the Englishman Newton and the German Leibniz immediately come to mind. These two outstanding scholars of the 17th century are accredited with the invention of calculus by most textbooks. It is most unfortunate that their concurrent but independent founding of their own schools of calculus led to an ugly plagiarism scandal, which resulted in a century of no exchange between British and continental mathematicians. Because of this, British mathematics did not advance for a century! Mathematicians must heed this historical lesson. The most regrettable aspect of this dispute was that it was actually instigated by an "outsider", the Swiss mathematician de Duillier, who held a grudge against Leibniz. As it intensified, the dispute became a display of nationalistic bigotry, and the protaganists could not escape involvement. In fact, earlier, Leibniz had very high esteem for Newton, and was quoted as saying that Newton's contribution to mathematics had been the better half of the sum of all before him. This dispute of precedence is absolutely pointless as numerous scholars had made contributions to the developement of calculus in the 17th century. It is fair to say that by the mid-17th century, all the groundwork had been prepared and the time was ripe for calculus to make its grand entrance; Newton and Leibniz compiled the knowledge and put on the final touches. This assessment does not in any way lessen the important contributions by these two great mathematicians. We must bear in mind what era they lived in. Some people have called the 17th century "the age of geniuses". The "geniuses" came about because they exceeded others in observational powers, knowledge, diligence and commitment. Thus they were able to make full use of the opportunities of their times to make the maximum contributions. Newton had said: "If I have seen further than [others], it is because I have stood on the shoulders of giants." His secret of success was that he never stopped thinking. It was usual for him to work 18–19 hour days and he had spent many sleepless nights in the laboratory.

Calculus without Newton and Leibniz is like the Trojan War without the wooden horse. In view of the fact that the groundwork had been laid before, what then really were the contributions of these two great men?

First, they put calculus into a systematic framework. According to Newton, his work on calculus was done during the period 1665–1666; however, distaste for publication led to the delay of its dissemination. His major publications include *De analysi per aequationes numero terminorum infinitas* (completed in 1669 but published in 1711), *Methodus fluxionum et serierum infinitorum* (completed in 1671 and published in 1742), *De quadratura curvarum* (completed in 1676 and published in 1693). His magnum opus, though, is his treatise *Philosophiae naturalis principia mathematica*, in which he expounded his theories of calculus as well as mechanics. This treatise was completed later than the three mentioned above but was the first in print (1687), thanks to his good friend and fellow mathematician Halley, who dug into his own pockets to see its publication through.

On the other hand, Leibniz "discovered" calculus independently in 1676, but published his findings earlier than Newton. His first paper on differentiation appeared in a German journal in 1684, followed in 1686 by a paper on integration. He had originally used the terms *Calculus Differentialis* and *Calculus Summatorius,* which the Swiss mathematician James Bernoulli later changed to *Calculus Integralis*. This is the origin of the terminology in English. Before Newton and Leibniz, calculus consisted of a collection of "clever tricks" proposed by many different people. It was Newton and Leibniz who organized them into a systematic, widely-applicable discipline with standardized notation. Before that, its use had been extemporaneous – every problem required a different "trick" and the hard work of many first-rate minds. We must thank Newton and Leibniz if only for this.

Their (especially Newton's) second main contribution was that they brought "infinity" into the realm of mathematics. Newton observed that the infinite sum could be viewed as an extension of ordinary algebraic operations. As an example, his celebrated Binomial Theorem was an extension of the binomial theorem in algebra. For a natural number N, it had been known that

$$(A + B)^N = \binom{N}{0} A^N + \binom{N}{1} A^{N-1} B + \ldots + \binom{N}{N-1} AB^{N-1} + \binom{N}{N} B^N, \qquad (2)$$

where $\binom{N}{r}$ denotes the binomial coefficient which is the number of ways one can choose r objects out of N different ones. For a number N which is not a natural number, Newton showed that

$$(A + B)^N = A^N + NA^{N-1}B + \frac{N(N-1)}{1 \cdot 2} A^{N-2}B^2 + \frac{N(N-1)(N-2)}{1 \cdot 2 \cdot 3} A^{N-3}B^3 + \ldots \qquad (3)$$

Of course, today we can see that the formula (3) is an obvious generalization of (2) by writing out the binomial coefficients. (Note that the illusion that the infinite sum (3) is merely a straightforward extension of the finite sum (2) is fallacious.) However, this is far from the way of thinking during Newton's time; he derived the formula (3) while considering an integration problem. On the other hand, the discovery of this formula allowed him to compute the derivative of x^n, and it was through this sort of connection that he finally came upon the Fundamental Theorem of Calculus. The incorporation of "infinity" into mathematics broke the Hellenic tradition of avoiding infinite processes and began a new chapter for mathematics.

In the subsequent two centuries, many advances in calculus had been made. Theorems upon theorems had been discovered, leading to the solution of one practical problem upon another. But still, the logical foundation of calculus was pitifully deficient. The anology is a barely-equipped commando team going deep into enemy territories, taking one stronghold after another, while its supplies and back-up are nowhere in sight. Most people find this phenomenon shocking as they believe that mathematics is a science in which logic and rigour reign supreme. The 18th century English mathematician de Morgan said: "The driving force of mathematical discoveries is not deduction but imagination." It was by the sheer courage of 18th and 19th century mathematicians that calculus became a profound and versatile body of knowledge. While there had been minor mistakes, it verged upon a miracle that, lacking a solid foundation, there had not been any major breakdown in more than two hundred years and that it had not gone astray. In addition to the extraordinary intelligence and foresight of the master mathematicians, this was due to the fact that, at the time, mathematics developed hand in hand with physics and astronomy. Mathematical discoveries were "empirically verified" by scientific experiments, and at the same time, problems originating in science prompted the development of mathematical theories. This partnership with science provided the strong dose of confidence that mathematics needed.

In 1734, the English Bishop Berkeley wrote *The Analyst* in which he attacked calculus for its lack of logical foundation. This tract was subtitled "Or a Discourse Addressed to an Infidel Mathematician [presumably Halley]. Wherein It Is Examined Whether the Object, Principles, and Inferences of the Modern Analysis are More Distinctly Conceived, or More Evidently Deduced, than Religious Mysteries and Points of Faith." From this, Bishop Berkeley's motivation for attacking calculus was clear. However, from the mathematical point of view, he did raise numerous issues which had to be addressed. As more and more such questions badly needed answers, mathematicians in the mid-19th century realized the imperiousness of building a rigorous foundation for calculus. Thus the need to give calculus a new lease on life gave birth to a vast subject in mathematics – mathematical analysis. M²