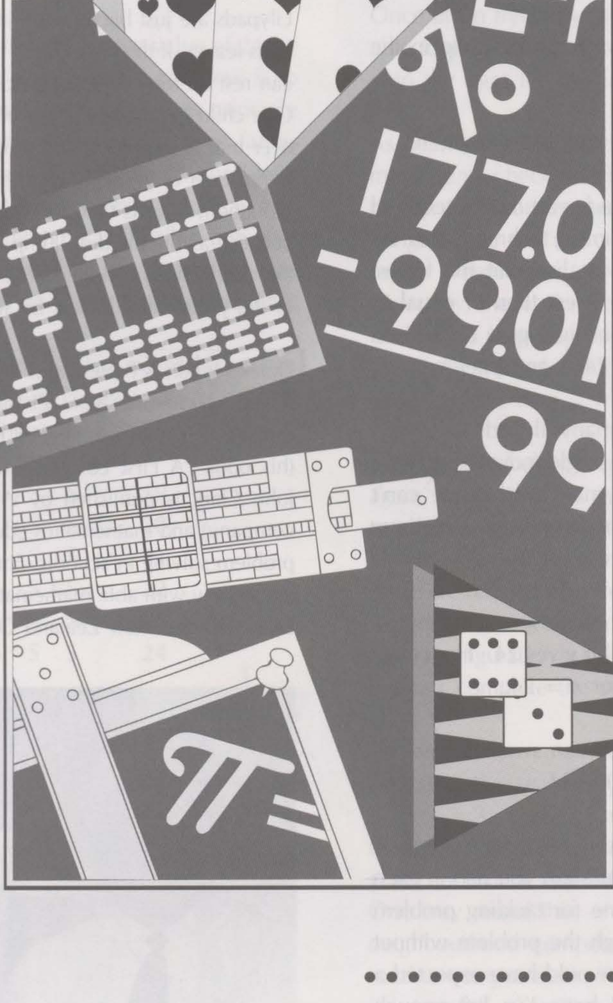
COUNTING
- Its Principles and Techniques (5) -

by K M Koh and B P Tan



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12. The Binomial Expansion

In Section 4 of [2], we introduced a family of numbers which was denoted by $\binom{n}{r}$ or C_r^n. Given any integers n and r with $0 \le r \le n$, the number $\binom{n}{r}$ is defined as the number of r-element subsets of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$. That is, $\binom{n}{r}$ is the number of ways of selecting r distinct objects from a set of n distinct objects. We also derived the following formula for $\binom{n}{r}$:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad (1)$$

By applying (1), or otherwise, we can easily derive some interesting identities involving these numbers such as

$$\binom{n}{r} = \binom{n}{n-r}, \qquad (2)$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \qquad (3)$$

$$r\binom{n}{r} = n\binom{n-1}{r-1}, \quad r \ge 1 \qquad (4)$$

$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}. \qquad (5)$$

In this article, we shall learn more about this family of numbers and derive some more important identities involving $\binom{n}{r}$.

Problem 12.1

Prove identities (4) and (5). (Note that (4) is a special case of (5).)

In algebra, we have learnt how to expand (assuming that the usual commutative, associative and distributive laws hold) the algebraic expression $(1 + x)^n$ for $n = 0, 1, 2, 3$. Their expansions are shown below:

$$(1 + x)^0 = 1$$
$$(1 + x)^1 = 1 + x$$
$$(1 + x)^2 = 1 + 2x + x^2$$
$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

Notice that the coefficients in the above expansions are actually numbers of the form $\binom{n}{r}$. Indeed, we have:

$$1 = \binom{0}{0}$$

$$1 = \binom{1}{0} \qquad 1 = \binom{1}{1}$$

$$1 = \binom{2}{0} \qquad 2 = \binom{2}{1} \qquad 1 = \binom{2}{2}$$

$$1 = \binom{3}{0} \qquad 3 = \binom{3}{1} \qquad 3 = \binom{3}{2} \qquad 1 = \binom{3}{3}.$$

What can we say about the coefficients in the expansion of $(1 + x)^4$? Would we obtain

$$(1 + x)^4 = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4 \;?$$

Let us try to find out the coefficient of x^2 in the expansion of $(1 + x)^4$. We may write

$$(1 + x)^4 = \underset{(1)}{(1 + x)}\underset{(2)}{(1 + x)}\underset{(3)}{(1 + x)}\underset{(4)}{(1 + x)}.$$

Observe that in the expansion, each of the factors (1), (2), (3) and (4) contributes either '1' or 'x', and they are multiplied to form a term. For instance, to obtain x^2 in the expansion, two of (1), (2), (3) and (4) contribute 'x' and the remaining two contribute '1'. How many ways can this be done? Table 1 shows all possible ways, and the answer is 6.

(1)	(2)	(3)	(4)
x	x		
x		x	
x			x
	x	x	
	x		x
		x	x

Table 1

Thus there are 6 terms of x^2 and the coefficient of x^2 is therefore 6 in the expansion of $(1 + x)^4$. Indeed, to select 2 'x' from 4 factors $(1 + x)$, there are $\binom{4}{2}$ ways (and the remaining two have no choice but contribute '1'). Thus the coefficient of x^2 in the expansion of $(1 + x)^4$ is $\binom{4}{2}$, which is '6'. Using a similar argument, one can readily see that

$$(1 + x)^4 = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \binom{4}{3}x^3 + \binom{4}{4}x^4 .$$

In general, what can be said about the expansion of $(1 + x)^n$, where n is any natural number?

Let us write

$$(1 + x)^n = \underset{(1)}{(1 + x)}\,\underset{(2)}{(1 + x)} \ldots \underset{(n)}{(1 + x)}. \qquad (*)$$

To expand $(1 + x)^n$, we first select either '1' or 'x' from each of the n factors $(1 + x)$, and then multiply the n chosen '1 or x' together. The general term thus obtained is of the form x^r, where $0 \le r \le n$. What is the coefficient of x^r in the expansion of $(1 + x)^n$ if the like terms are grouped? This coefficient is the number of ways to form the term 'x^r' in the product (*). To form a term 'x^r', we choose r factors $(1 + x)$ from the n factors $(1 + x)$ in (*) and select 'x' from each of the r factors chosen. Each of the remaining $n - r$ factors $(1 + x)$ has no choice but contributes '1'. Clearly, the above selection can be done in $\binom{n}{r}$ ways. Thus the coefficient of x^r in the expansion of $(1 + x)^n$ is given by $\binom{n}{r}$. We thus arrive at the following result which was first discovered by Newton:

The Binomial Theorem (BT)

For any natural number n,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n}x^n .$$

As $\binom{n}{r}$'s are the coefficients of the terms in the expansion of $(1+x)^n$, these numbers are often called the *binomial coefficients*.

13. Some Useful Identities

We gave four simple identities involving binomial coefficients, namely (2) - (5), in the above section. In this section, we shall derive some more identities involving binomial coefficients from (BT). These identities are not only interesting in their own right, but also useful in simplifying certain algebraic expressions.

Consider the expansion of $(1+x)^n$ in (BT). If we let $x = 1$, we then obtain from (BT) the following:

(B1) $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n .$$

Example 13.1

In Example 6.2 of [3], we discussed a counting problem on $\wp(S)$, the set of all subsets of a finite set S. It can be shown by applying (BP) (see Problem 6.1 of [3]) that if S is an n-element set (i.e., $|S| = n$), then there are exactly 2^n subsets of S inclusive of the empty set ϕ and the set S itself (i.e., $|\wp(S)| = 2^n$). We can now give a more 'natural' proof for this fact. Assume that $|S| = n$. By definition, the number of

0 - element subset of S is $\binom{n}{0}$,

1 - element subsets of S is $\binom{n}{1}$,

2 - element subsets of S is $\binom{n}{2}$,

n - element subset of S is $\binom{n}{n}$.

Thus,

$$|\wp(S)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$$

$$= 2^n \qquad \text{(by (B1))}.$$

Example 13.2

The number '4' can be expressed as a sum of one or more positive integers, taking order into account, in the following 8 ways:

$$\begin{aligned} 4 &= 4 \\ &= 1 + 3 \\ &= 3 + 1 \\ &= 2 + 2 \\ &= 1 + 1 + 2 \\ &= 1 + 2 + 1 \\ &= 2 + 1 + 1 \\ &= 1 + 1 + 1 + 1. \end{aligned}$$

Show that every natural number n can be so expressed in 2^{n-1} ways.

This is in fact Problem 6.5 stated in [3]. Let us see how (B1) can be used to prove the result. But first of all, consider the above special case when $n = 4$.

We write $4 = 1 + 1 + 1 + 1$ and note that there are three '+''s in the expression. Look at the following relation.

$$\begin{aligned} 4 &\leftrightarrow \underbrace{1 + 1 + 1 + 1}_{4} \\ 1 + 3 &\leftrightarrow \underbrace{1}_{1} \oplus \underbrace{1 + 1 + 1}_{3} \\ 3 + 1 &\leftrightarrow \underbrace{1 + 1 + 1}_{3} \oplus \underbrace{1}_{1} \\ 2 + 2 &\leftrightarrow \underbrace{1 + 1}_{2} \oplus \underbrace{1 + 1}_{2} \\ 1 + 1 + 2 &\leftrightarrow \underbrace{1}_{1} \oplus \underbrace{1}_{1} \oplus \underbrace{1 + 1}_{2} \\ 1 + 2 + 1 &\leftrightarrow \underbrace{1}_{1} \oplus \underbrace{1 + 1}_{2} \oplus \underbrace{1}_{1} \\ 2 + 1 + 1 &\leftrightarrow \underbrace{1 + 1}_{2} \oplus \underbrace{1}_{1} \oplus \underbrace{1}_{1} \\ 1 + 1 + 1 + 1 &\leftrightarrow \underbrace{1}_{1} \oplus \underbrace{1}_{1} \oplus \underbrace{1}_{1} \oplus \underbrace{1}_{1} \end{aligned}$$

This relation is actually a bijection between the set of all such expressions of '4' and the set of all subsets of three '+''s. Thus, by (BP) and (BT), the required answer is

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$$

In general, write

$$n = \underbrace{1 + 1 + \dots + 1 + 1}_{n}$$

and note that there are $n - 1$ '+''s in the expression. By extending the above technique of establishing a bijection, it can be shown that the number of all such expressions of 'n' is

$$\binom{n-1}{0} + \binom{n-1}{1} + \dots + \binom{n-1}{n-1} = 2^{n-1} \qquad \text{(by B(1))}.$$

Problem 13.1

By applying identity (5), or otherwise, show that

$$\sum_{k=r}^{n} \binom{n}{k}\binom{k}{r} = \binom{n}{r} 2^{n-r},$$

where $0 \le r \le n$.

Problem 13.2

Show that

$$\sum_{k=0}^{n-1} \binom{2n-1}{k} = 2^{2n-2}.$$

Consider again the expansion of $(1 + x)^n$ in (BT). If we now let $x = -1$ we then have

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n\binom{n}{n} = 0,$$

where the terms on the LHS alternate in sign. Thus, if n is even, say $n = 2k$, then

$$\binom{n}{0} + \binom{n}{2} + \ldots + \binom{n}{2k} = \binom{n}{1} + \binom{n}{3} + \ldots + \binom{n}{2k-1};$$

and if n is odd, say $n = 2k + 1$, then

$$\binom{n}{0} + \binom{n}{2} + \ldots + \binom{n}{2k} = \binom{n}{1} + \binom{n}{3} + \ldots + \binom{n}{2k+1}.$$

As

$$\left[\binom{n}{0} + \binom{n}{2} + \ldots\right] + \left[\binom{n}{1} + \binom{n}{3} + \ldots\right] = 2^n$$

by (B1), we have

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

$$= \frac{1}{2}(2^n) = 2^{n-1}. \qquad \text{(B2)}$$

Example 13.3

A finite set S is said to be *even* (resp., *odd*) if $|S|$ is even (resp., odd). Consider $\mathbb{N}_8 = \{1, 2, \ldots, 8\}$. How many even subsets does $\mathbb{N}_8$ have? How many odd subsets does $\mathbb{N}_8$ have?

The number of even subsets of $\mathbb{N}_8$ is

$$\binom{8}{0} + \binom{8}{2} + \binom{8}{4} + \binom{8}{6} + \binom{8}{8},$$

and the number of odd subsets of $\mathbb{N}_8$ is

$$\binom{8}{1} + \binom{8}{3} + \binom{8}{5} + \binom{8}{7}.$$

By (B2),

$$\binom{8}{0} + \binom{8}{2} + \ldots + \binom{8}{8} = \binom{8}{1} + \binom{8}{3} + \binom{8}{5} + \binom{8}{7}$$

$$= 2^{8-1} = 2^7 = 128.$$

Problem 13.3

By applying identity (4), or otherwise, show that

$$\sum_{k=1}^{n} (-1)^k k\binom{n}{k} = 0 .$$

Consider the following binomial expansion once more:

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n .$$

If we treat the expressions on both sides as functions of x, and differentiate them with respect to x, we obtain:

$$n(1 + x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \ldots + n\binom{n}{n}x^{n-1} .$$

By letting $x = 1$ in the above identity, we have:

$$\text{(B3)} \quad \sum_{k=1}^{n} k\binom{n}{k} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \ldots + n\binom{n}{n}$$
$$= n \cdot 2^{n-1} .$$

Note also that (B3) is a special case of the identity stated in Problem 13.1 when $r = 1$.

Let us try to derive (B3) by a different way. Consider the following problem. Suppose that there are $n\,(n \geq 1)$ people in a group, and they wish to form a committee consisting of people from the group, including the selection of a leader for the committee. In how many ways can this be done?

Let us illustrate the case when $n = 3$. Suppose that A, B, C are the 3 people in the group, and that a committee consists of k members from the group, where $1 \leq k \leq 3$. For $k = 1$, there are 3 ways to do so as shown below

Committee member	*Leader*
A	A
B	B
C	C

For $k = 2$, there are 6 ways to do so as shown below.

Committee members	*Leader*
A, B	A
A, B	B
A, C	A
A, C	C
B, C	B
B, C	C

For $k = 3$, there are 3 ways to do so as shown below.

Committee members	*Leader*
A, B, C	A
A, B, C	B
A, B, C	C

Thus, there are altogether $3 + 6 + 3 = 12$ ways to do so.

In general, from a group of n people, there are $\binom{n}{k}$ ways to form a k-member committee, and k ways to select a leader from the k members in the committee. Thus, the number of ways to form a k-member committee including the selection of a leader is, by (MP), $k\binom{n}{k}$. As k could be 1, 2, ..., n, by (AP), the number of ways to do so is given by

$$\sum_{k=1}^{n} k\binom{n}{k}.$$

Let us count the same problem with a different approach as follows. First, we select a leader from the group, and then choose other $k - 1$ members ($k = 1, 2, ..., n$) from the group to form a k-member committee. There are n choices for the first step and $\binom{n-1}{0} + \binom{n-1}{1} + ... + \binom{n-1}{n-1}$ ways for the second step. Thus, by (MP) and (B1), the required number is

$$n\left[\binom{n-1}{0} + \binom{n-1}{1} + ... + \binom{n-1}{n-1}\right] = n\cdot 2^{n-1}$$

Since both $\sum_{k=1}^{n} k\binom{n}{k}$ and $n\cdot 2^{n-1}$ count the same number, identity (B3) follows.

In the above discussion, we establish identity (B3) by first introducing a 'suitable' counting problem, we then count the problem in 2 different ways so as to obtain 2 different expressions. These 2 different expressions must be equal as they count the same quantity. This way of deriving an identity is quite a common practice in combinatorics, and is known as 'Counting it twice'.

Problem 13.4.

Show that

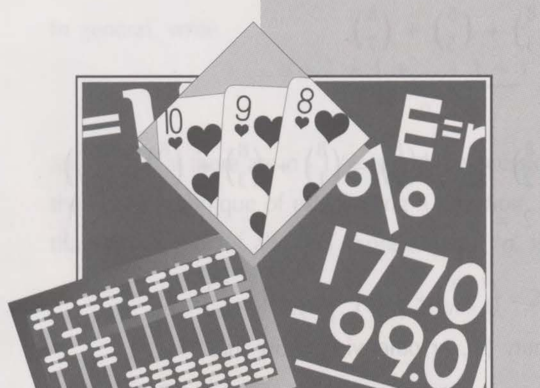
$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}(2^{n+1} - 1)$$

by integrating both sides of $(1 + x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ with respect to x.

Problem 13.5

Show that

$$\sum_{k=1}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}.$$



References

[1] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (1)*, Mathematical Medley Vol 22 March (1995) 8-13.

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[3] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (3)*, Mathematical Medley Vol 23 March (1996) 9-14.

[4] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (4)*, Mathematical Medley Vol 23 September (1996) 44-50.