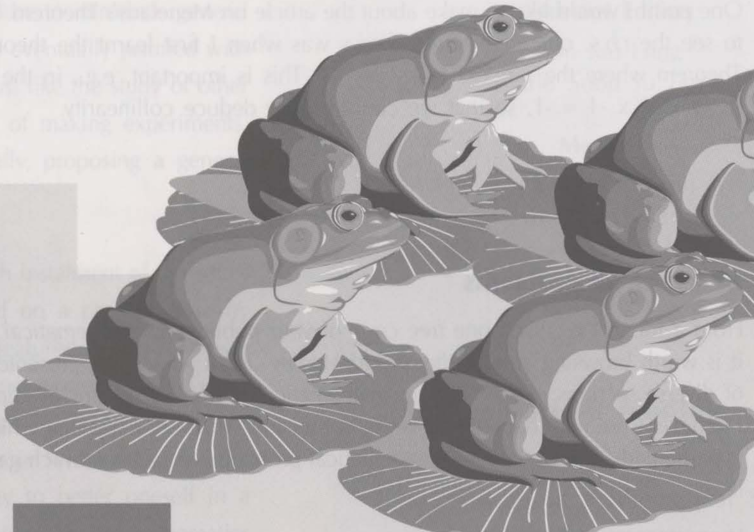
Problem Solving Frogs



by Derek Holton

Let's start with a problem. I have four green frogs on four lilypads on the right and four spotted frogs on four lilypads on the left. There is one lilypad in the middle between them. The green frogs want to go to the left and the spotted frogs want to go to the right. They can move by hopping on to an adjacent empty lilypad or by jumping over one other frog onto an adjacent empty lilypad. What is the smallest number of moves which will get the green frogs to the left and the spotted frogs to the right?

The initial and final positions of the frogs is shown below in Figure (a) and (b) respectively.

(a) S S S S _ G G G G

(b) G G G G _ S S S S

Figure 1

At this stage in almost any problem, I don't have a clue as to what to do. If anyone is looking over my shoulder, then I generally panic. After a little while though, it seems to me that it might be useful if I could get hold of some frogs. Plastic ones are fine. Real ones tend not to jump the way you want them to. But pieces of paper will do. Then it's experiment time. Just play with your frogs and hope you get a good idea.

The first good idea seems to be that eight frogs are too much to handle. I can't keep track of them all. If I play around with just one or two, then that might give me the breakthrough I need. Trying a **single case** is always a useful problem solving strategy. Let's start with one green frog and one spotted frog. Figure 2 shows that I can swap single frogs around easily.

S _ G

S G _ 1

_ G S 2

G _ S 3

One frog a side in 3 moves.

Figure 2

I'm not sure that that told me too much though. It was far too easy. Maybe two frogs a side will be better. I've made a start in Figure 3.

S S _ G G

S S G _ G 1

S S G G _ 2

Figure 3

The second move in Figure 3 doesn't seem too smart. The only two moves I have available send me back to a previous position. It looks as if I should have moved a spotted frog on my second go rather than a green one. Let's see how far that gets me.

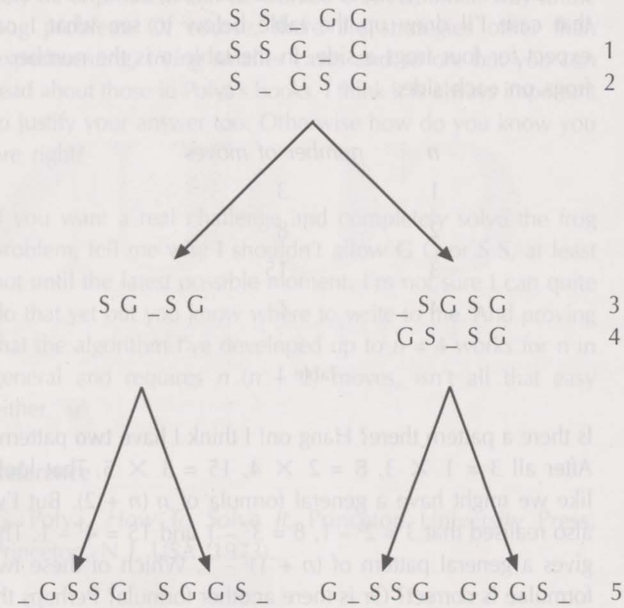


Figure 4

Figure 4 shows various directions I could go, assuming all frogs always move forward. Surely backward moves will mean more moves, so it's a fair guess that I should keep going forward if at all possible.

Three of the bottom four situations of Figure 4 worry me. In two of them I can make one more forward move before I have to send at least one frog back. The third position forces a backward move straight away. For the moment I'll keep going with the situation on the extreme right because this looks more promising. This is done in Figure 5.

G S G S _ 5

G S G _ S 6

G _ G S S 7

G G _ S S 8

Two frogs a side in 8 moves

Figure 5

So I can swap two pairs of frogs in 8 moves. I think it's not too difficult to show that it can't be done in fewer moves. Certainly the situations that caused backward moves lead to more than 8 moves.

Before I go on to three frogs it's worth reflecting. What are good positions to get into and what are bad positions? I seem to have to back up when I get two frogs of the same type together. Is this what I need to avoid? We'll need to think about that for a bit. Clearly at the beginning and towards the end, we want two (or more) of the same frogs to be next to each other. Along the way though they appear to be a bad thing. Let's keep this in mind and do the three frogs case.

Ah! Keeping pairs of the same type of frog apart for as long as possible, I can swap them over in 15 moves. Can you do any better? I'm going to assume that you can't for now. In that case I'll draw up the table below to see what I can expect for four frogs a side. In the table n is the number of frogs on each side.

n	*number of moves*
1	3
2	8
3	15
4	?

Table 1

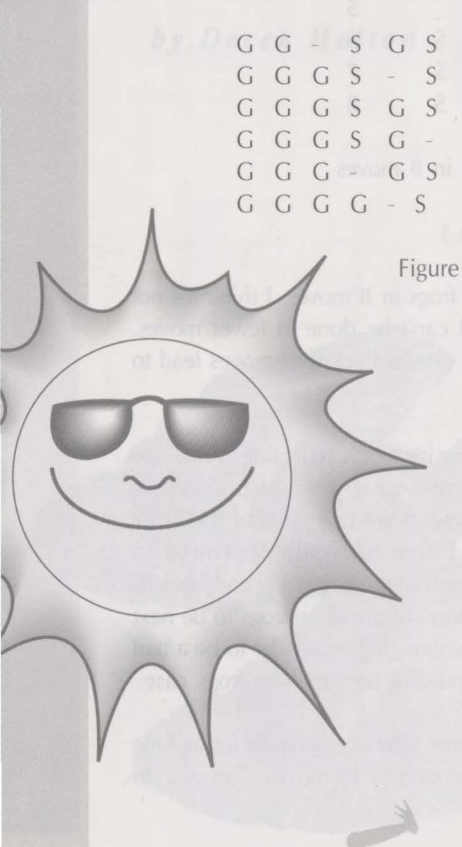
Is there a pattern there? Hang on! I think I have **two** patterns. After all $3 = 1 \times 3$, $8 = 2 \times 4$, $15 = 3 \times 5$. That looks like we might have a general formula of $n(n + 2)$. But I've also realised that $3 = 2^2 - 1$, $8 = 3^2 - 1$ and $15 = 4^2 - 1$. This gives a general pattern of $(n + 1)^2 - 1$. Which of these two formulae is correct? Or is there another formula? Perhaps the correct one.

Actually both the formulae I've found give the value of 24 for the 4 green and 4 spotted frog problem. The test now is to see whether I can get that or not. But the guiding principle I'll use is to keep frogs of the same type apart as far as possible and use the ideas of the two and three frog problem. Actually the only other idea that seems to exist is to leap frog forward as many frogs of the same type as possible one after the other. In between times, the odd frog has to slide forward just one lilypad in order to open up a string of leap frogging.

OK. So I think I'm on top of that. With a bit of luck I can probably work it out. So let's do it. Figure 6 shows the last few moves. I'll let you fill in the early details.

G	G	-	S	G	S	G	S	S	19
G	G	G	S	-	S	G	S	S	20
G	G	G	S	G	S	-	S	S	21
G	G	G	S	G	-	S	S	S	22
G	G	G	-	G	S	S	S	S	23
G	G	G	G	-	S	S	S	S	24

Figure 6



The question is, have we now finished? We've certainly managed to swap our frogs in 24 moves but that wasn't the original question. That asked for the **smallest** number of moves required. How do we know that 24 is the **smallest**?

There are various ways that you can react to this. One of them is to say "Well, I enjoyed playing with the frogs and I don't really care if it's not the smallest". Another way to react is to say "I'm blowed if I can see any other way of doing things so this must be the smallest." Unfortunately mathematicians, bless their cotton socks, will say "There must be a proof, if it's true, or a counterexample, if it's false, I won't rest until I've found one or the other".

Now mathematicians can be a pain at times but I'm going to follow through with the last question. If 24 is right, we might be able to find a proof. If 24 is wrong, then there has to be another way to do the swapping which uses **less** than 24 moves. This would be the counterexample. So how can I find a proof or a counterexample?

Once again trying the 4 case looks too hard. But I might be able to prove things for $n = 1$, 2, and 3. That might put me onto the right track for $n = 4$.

As before, $n = 1$ is a doddle. I can put down all possible moves and check out that I get the swap first after three moves. An **exhaustive search** does the problem here. To speed things up we can use a bit of **symmetry**. After all, it doesn't matter whether we move G or S first. So, without loss of generality, move G (as I did in Figure 2.) The second move is clearly a leap frog by S (otherwise you get back to where you started). Moving G completes the swap. Clearly we can't do any better than 3 with $n = 1$.

Let's keep the ball rolling for $n = 2$. Without loss of generality, again, move the left most G. (Jumping a G puts two Gs together, which is probably bad.) Then we can go back to Figure 4. As we said before three of the situations at step 5 mean we require a backward move. Even if we do this move, we get back to a situation from a previous step. So the right hand configuration is the only active one. It's easy to see that we can complete the swap in less than three moves from here.

An exhaustive search, trying every case, works for $n = 2$. Try it for $n = 3$. It should work there too. By the time you get to $n = 4$ you should be convinced that 24 is best but it's going to take a tour de force to go through all the cases. One way round this, of course, is to bring in your friendly neighbourhood computer. These things are good at doing routine computations. What is an exhaustive search here if it is not full of routine computations?

Is there a better way though? This all reminds me of the Four Colour Theorem, where a computer proof is the only one we've got. But it never seems very satisfactory. How else can we settle this problem? Can we tackle it analytically in some way?

Let's go back to $n = 1$. How many lilypads do the frogs have to travel over or on? Well the green frog has to get by 2 and the spotted frog 2. Altogether 4 lilypads have to be traversed.

How many leap frogs are there? Surely only one. The green frogs leap the spotted or the spotted frogs leap the green. Once it's done, and everything keeps moving forward, then no frogs ever encounter each other again. All other moves must be simple moves onto a neighbouring lilypad. Now each leapfrog takes 2 lilypads so we've got $4 - 2 = 2$ more lilypads to traverse. As these need single moves there have to be 2 single moves. Now 2 single moves plus 1 leapfrog give the 3 moves we already found.

Let's try this for $n = 2$. Here the frogs have to traverse 3 lilypads each. So 12 lilypads have to be traversed in total. The only leap frogs occur when green frogs and spotted frogs meet. Each green frog meets 2 spotted frogs so there are 4 leapfrogs. These account for 8 lilypads. Since $12 - 8 = 4$, there must be 4 single moves. Now 4 single moves plus 4 leapfrogs gives a total of 8 moves.

We're cooking on gas! The case $n = 3$ can be done in the same way. Try it. I'll wait while you do.

If you've mastered that, then $n = 4$ should be a cinch!

If we are going to prove that 24 is the fewest moves required for $n = 4$, then before we use the counting technique that we invented for $n = 1, 2$, we have to realise that the fewest number of moves in the $n = 4$ case is **less than or equal to** 24. I can do it in 24 (see the end part in Figure 6) but one of you may see a quicker way. (So 24 is an upper bound.)

Now the counting technique. (How many lilypads traversed? How many leapfrogs? So how many single moves?) will give a lower bound to the problem. It may be that we can't actually use precisely the number of moves this technique gives us. However, we know that we must use at **least** this many moves.

Fortunately the counting technique also gives 24. Hence we know that 24 is the fewest moves for $n = 24$.

Now prove that n frogs on each side can swap themselves in $n(n + 2)$ moves. Or is it $(n + 1)^2 - 1$?

So how did you get on that? I couldn't talk to each one of you individually and answer every question you had at each stage. So this article is only a outline for tackling problem solving. I've tried to take you through the problem without telling you how to do it. In person I would have expected a bigger contribution from you. But I hope I've left enough open space for you to get some achievement out of the frog problem.

Any other problem can be worked on in a similar way to the frog problem. Of course, there are strategies other than experimenting, trying smaller cases and so on, but you can read about those in Polya's books. I think it is always important to justify your answer too. Otherwise how do you know you are right?

If you want a real challenge and completely solve the frog problem, tell me why I shouldn't allow G G or S S, at least not until the latest possible moment. I'm not sure I can quite do that yet but you know where to write to me. And proving that the algorithm I've developed up to $n = 4$ works for n in general and requires $n(n + 2)$ moves, isn't all that easy either. M^2

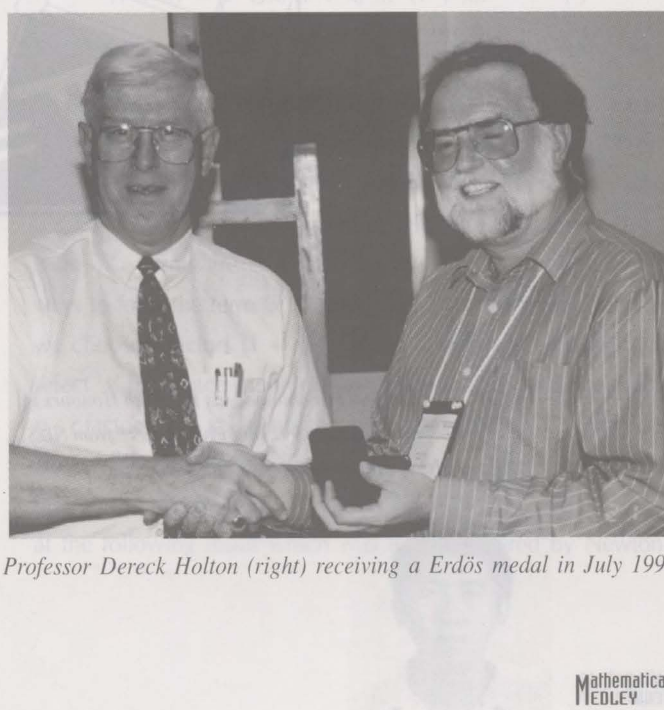
Reference

G. Polya, *How To Solve It*, Princeton University Press, Princeton, N J, USA (1973).

Author's note:

Lilypads are just big leaves that float on top of the water a bit like lotus leaves. If the lilypads are big enough frogs and other creatures can rest on top of them. Leap frog is a game played by children. One child bends over at the waist and the next one in line jumps over him. The second child then bends over. The third child then jumps over both bending children in turn. The game continues by induction. I guess that the frog part of the leap frog comes about because as the children jump they put they hands on the back of the child who is bending. With their arms in and their leags spread in the jump, they look a bit like frogs.

Derek Holton is Professor of Pure Mathematics at the University of Otago, Dunedin, New Zealand (possibly the most southerly university in the world). His research interests are graph theory (his book "A First Look at Graph Theory" authored jointly with John Clark is published by World Scientific, a Singapore based company) and mathematics education, specifically the teaching of problem solving in schools. For a number of years he has worked extensively with able mathematics sutdents and has been the Team Leader for the New Zealand Olympiad team on seven occasions.



Professor Dereck Holton (right) receiving a Erdös medal in July 1996