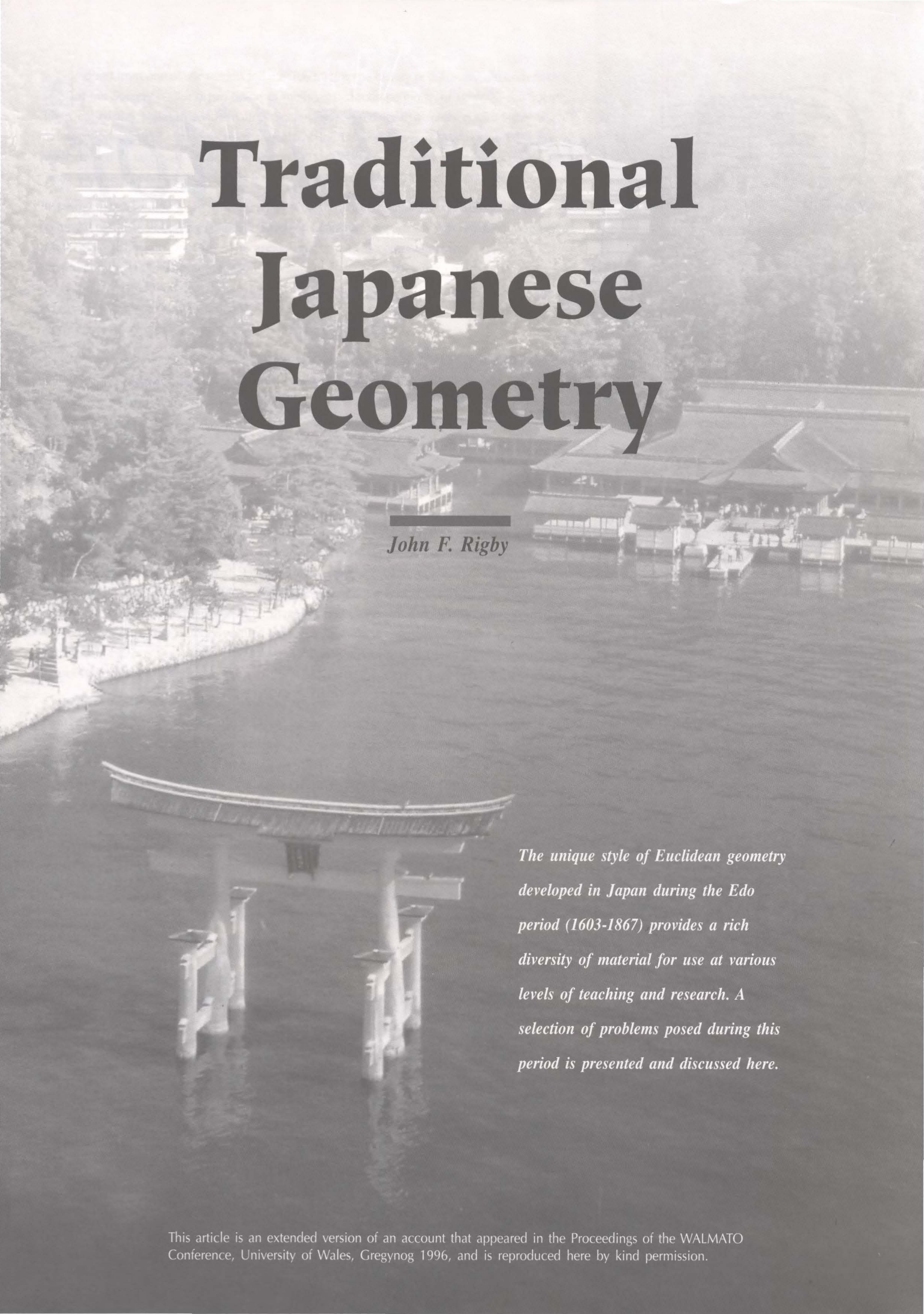

# Traditional Japanese Geometry

*John F. Rigby*

*The unique style of Euclidean geometry developed in Japan during the Edo period (1603-1867) provides a rich diversity of material for use at various levels of teaching and research. A selection of problems posed during this period is presented and discussed here.*

This article is an extended version of an account that appeared in the Proceedings of the WALMATO Conference, University of Wales, Gregynog 1996, and is reproduced here by kind permission.

In the opening paragraph of their book *Japanese Temple Geometry Problems* [1], H. Fukagawa and D. Pedoe write "During the greater part of the Edo period (1603–1867) Japan was almost completely cut off from the western world. Books on mathematics, if they entered Japan at all, must have been scarce, and yet, during this long period of isolation people of all social classes, from farmers to samurai, produced theorems in Euclidean geometry which are remarkably different from those produced in the West during the centuries of schism, and sometimes predated these theorems by many years."

As far as I know, [1] is the only extensive publication in English on the subject of Japanese geometry, and this book and the many books on the subject in Japanese consist mainly of collections of problems, with or without solutions. Therefore there are many questions to which I still do not know the answers. How much Western mathematics came to Japan before 1600, from where, and in what form? Did the Japanese have books of mathematical theory and techniques? The authors of [1] say that there were few colleges or universities in Japan during the period in question, but many private schools.

The problems, mainly geometrical, were either published in books or painted on wooden tablets containing text and coloured figures and hung in shrines and temples; hence the use of the term "temple geometry" (a term not used in Japan) although twice as many tablets appear in shrines as in temples. The word for Japanese mathematics is *wasan*, and the tablets are called *sangaku*. Collections of sangaku problems, with solutions, appeared in Japanese books in the 18th and 19th centuries, and there are modern books of collections of old problems.

It is my impression that this Japanese geometry differs from Western geometry mainly in the type of problem that seems to have interested and intrigued the Japanese geometers. The Japanese love of artistic design is evident in many of the figures.

Much of my information about the subject has been gained from Hiroshi Okumura, an associate professor at the Maebashi College of Technology, in Gunma Prefecture. We correspond regularly and have discussions at conferences. A few years ago he gave me a copy of *The Sangaku in Gunma* [2], published in 1987 by the Gunma Wasan Study Association of which he is a member. This book is a limited edition, beautifully printed, bound and illustrated. It is sometimes possible to understand a problem from its illustration without being able to read the Japanese text.

The text of wasan was usually written in *Kanbun*, which is based on Chinese (since the Japanese language was written mainly in Chinese characters, I presume this means that Kanbun uses Chinese, rather than Japanese, grammar and syntax). Kanbun cannot be read by most people in modern Japan.

Many of the books on wasan, including [1], are like sets of "Miscellaneous Exercises" at the end of a geometry textbook, but with the rest of the textbook missing. I propose here to give a few examples of such exercises of varying degrees of difficulty, to provide a flavour of the types of problem encountered in wasan. I am not giving solutions here, but my comments may be helpful. Some of the problems have been used in secondary school mathematics clubs, or have provided problems or articles in pedagogical journals.

**1. Given the radius of the large semicircle in figure 1, find the radii of the other circles and semicircles [2, p.67].**

If the radius of the large circle is 6, the other radii are 3, 2 and 1. Once the solution has been obtained, we observe that the centres of the circles and semicircles form various 3–4–5 triangles, and the figure can be built up by drawing the small circles first, then circumscribing the large circle.

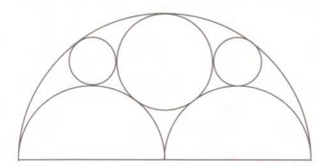

Figure 1

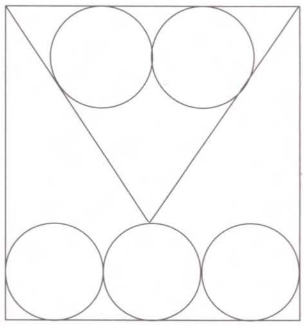

Figure 2

**2. In figure 2, which is bounded by a square, show that the two upper circles have the same radius as the three lower circles.**

This problem and the next were found by Hiroshi Okumura in a book of wasan problems in the Library of Congress, Washington [3]. If the square has side 6, once again we have a 3–4–5 triangle, and we have to prove that the radius of its inscribed circle is 1. This fact is not well known. My original proof used areas: the triangle $IBC$ is figure 3 has base $BC$ and height $r$ so its area is $BC.r/2$. Similarly the areas of $ICA$ and $IAB$ are $CA.r/2$ and $AB.r/2$. But the area of the right angled triangle $ABC$ is $BC.CA/2$; hence

$$BC.CA = (BC + CA + AB)r. \qquad (1)$$

Putting $BC = 3$, $CA = 4$, $AB = 5$ we deduce that $r = 1$.

A colleague of mine suggested a simpler proof. The tangents from $A$ to the incircle have equal length $x$ say, and the tangents from $B$ have equal length $y$ say; because of the right angle at $C$ the tangents from $C$ have length $r$. Putting $y + r = 3$, $r + x = 4$, $x + y = 5$ we deduce that $r = 1$.

Suppose now that figure 3 shows a *general* right angled triangle. Putting $BC = y + r$, $CA = r + x$, $AB = x + y$ in (1) we obtain

$$r^2 + rx + ry + xy = (2r + 2x + 2y)r;$$

if we multiply by 2, this can then be rewritten as

$$(r + x)^2 + (r + y)^2 = (x + y)^2,$$

and so we have a proof of Pythagoras' theorem using the incircle, which may not be new but I have not seen it before.

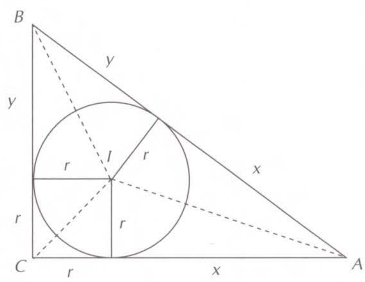


Figure 3

Finally on this topic, suppose the sides of a right angled triangle have *integer* lengths $a = y + r$, $b = r + x$, $c = x + y$; then $a^2 + b^2 = c^2$ so it is easy to prove either $a$, $b$, $c$ are all even or only one of them is even. Hence $r = (a + b - c)/2$ is an integer.

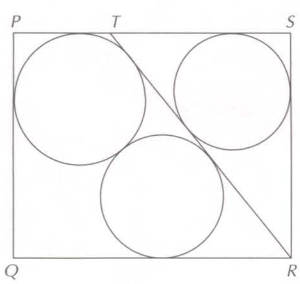


Figure 4

**3. In figure 4, all three circles inside the rectangle have equal radius. Show that the lengths *RQ* and *RT* are equal.**

We can also ask: what is the ratio of the sides of the rectangle? The figure has been drawn inaccurately so as not to reveal the answer. This problem can be generalised, as stated below, to the case where the interior line does not pass through the vertex $R$.

**4. In figure 5, all three circles inside the rectangle have equal radius. Show that the lengths *RQ* and *UT* are equal.**

For a solution of problems 3 and 4, see [5].

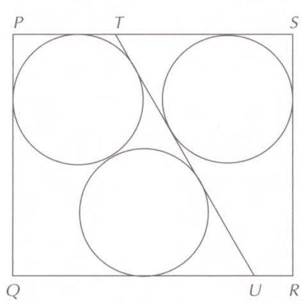


Figure 5

**5. In figure 6, two circles of equal radius *a* touch the outer circle (of radius *r*) at opposite ends of a diameter, and the same is true of the circles of radii *b* and *c*. Show that $a + b + c = r$.**

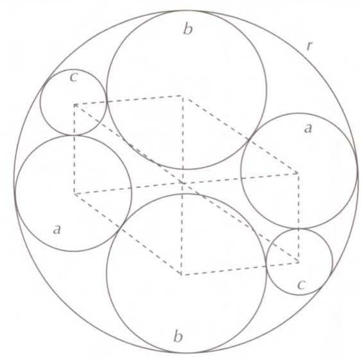

Figure 6

This problem, with a long solution, appears in a collection of wasan problems [8]. Hiroshi Okumura has found a simple solution by observing that the centres of the seven circles form the vertices of six congruent triangles with sides of lengths $b + c$, $c + a$, $a + b$. As in problem 1, the smaller circles can be drawn first, with a final circle of radius *r* circumscribing them [4].

Many wasan problems involve touching circles, as a glance at the figures in [1] and [2] will show, but many do not. Before giving examples of other types of problem, it is worth mentioning two well known results. A problem on a tablet from 1796, since lost, asked for a formula connecting the radii of four circles each of which touches the other three [1, p. 90]; such a formula was given earlier by Descartes. The corresponding problem for five spheres, each of which touches the other four, appeared on a tablet in 1785, and a solution was given in a woodblock-printed book in 1841. This solution required six pages: the powerful technique of inversion was apparently not known in Japan at the time [1, p.160, pp.179-186].

**6. In figure 7, which is based on five squares, show that the area *T* of the triangle and the area *S* of the square are equal [1, p.48].**

There must be tedious ways of proving this using excessive trigonometry or coordinate geometry, but it is a challenge to find a solution that is visually clear. The problem, with my solution, appears in [7].

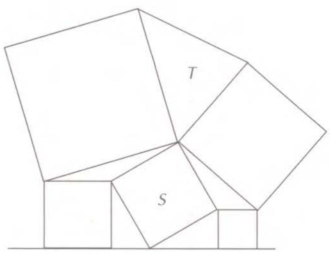

Figure 7

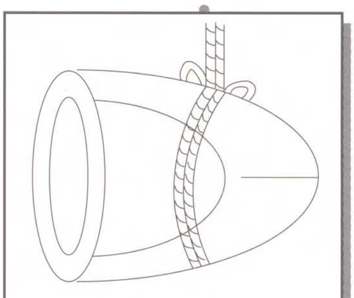
Figure 8

**7. In figure 8, find the centre of gravity of the solid semi-ellipsoid of revolution, formed from two ellipses with common major and minor axes.**

The figure comes from [2, p.157], and is redrawn by kind permission; my original guess, that it shows half a coconut hung up for birds to feed on (a common practice in the U.K.), was not far short of the mark!

**8. In figure 9 a rectangle of paper is folded so that opposite corners coincide. If the longer side of the rectangle is of given fixed length, what shape of rectangle gives the greatest value of the area of *ABC*?**

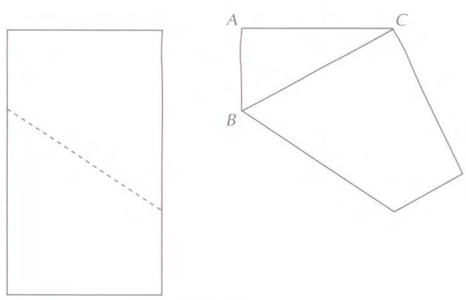

Figure 9

This problem comes from [2, p.188]. The original figure merely shows two folded sheets of paper, red on the inside and white on the outside, each inserted in the top of a wooden stick which is placed in a small jar of rice wine; the two jars rest on a table. The problem cannot be deduced from this figure without a translation of the text!

The solution of the previous problem requires elementary calculus. A remark in [1, p.138] mentions the only known reference to a parabola, from 1844, in which calculus is involved, so the methods of calculus must have been available. There is also only one reference to a hyperbola; this is apparently because the Japanese regarded an ellipse as a section of a right circular cylinder rather than of a cone, and a hyperbola cannot be obtained in this way. But problems involving ellipses abound. The next example is so simple and elegant that I am surprised that I have not found it in any western text on conic sections.

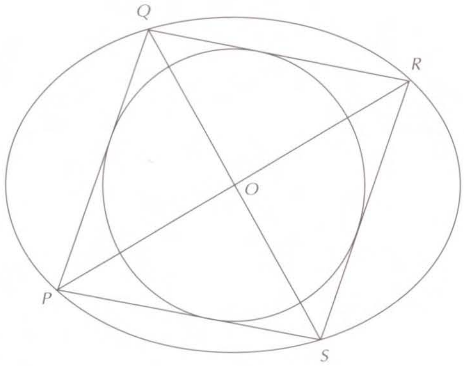


Figure 10

**9. Let *POR* and *QOS* be perpendicular diameters of an ellipse, as in figure 10. Then *PSRQ* is a rhombus, which has an inscribed circle with centre *O*. Show that the radius of this circle is independent of the two diameters (1841); [1, p.68].**

The solution given in [1, p.159], using coordinate geometry, can be shortened. We make use of a lemma that is also of interest in its own right.

**Lemma**

In figure 11 suppose the axes of the ellipse have lengths $2a$ and $2b$, and suppose the perpendicular diameters *POR* and *QOS* have lengths $2r$ and $2s$. Then

$$r^{-2} + s^{-2} = a^{-2} + b^{-2}.$$

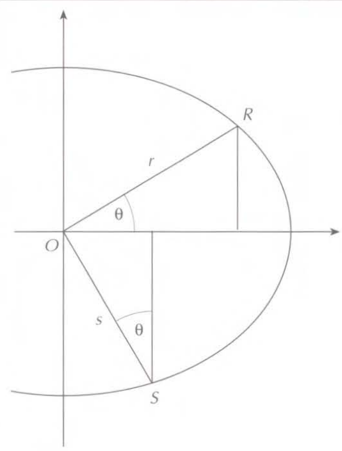


Figure 11

**Proof**

If we use the axes of the ellipse as coordinate axes, its equation is

$$x^2/a^2 + y^2/b^2 = 1.$$

The points $R$ and $S$ with coordinates

$(r \cos \theta, r \sin \theta)$ and $(s \sin \theta, -s \cos \theta)$

lie on the ellipse; hence $r^2\cos^2\theta /a^2 + r^2\sin^2\theta /b^2 = 1$

so that $1/r^2 = \cos^2\theta /a^2 + \sin^2\theta /b^2$;

similarly $1/s^2 = \sin^2\theta /a^2 + \cos^2\theta /b^2$.

Hence by addition $r^{-2} + s^{-2} = a^{-2} + b^{-2}$.

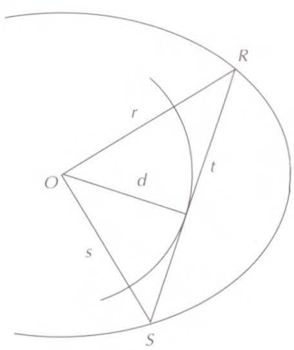


Figure 12

Now in figure 12 let the radius of the inscribed circle be $d$, and write $RS = t$. The area of triangle *ROS* can be calculated in two ways as $rs/2$ or $td/2$. Hence $r^2s^2 = t^2d^2 = (r^2 + s^2)d^2$. Hence $d^{-2} = r^{-2} + s^{-2} = a^{-2} + b^{-2}$ by the lemma. So the radius $d$ is independent of the diameters *POR* and *QOS*.

Note that the corresponding results $d^{-2} = r^{-2} + s^{-2} = a^{-2} - b^{-2}$ remain true for the hyperbola $x^2/a^2 - y^2/b^2 = 1$ as long as $b^2 > a^2$.

The next two problems were brought to my attention by Hiroshi Okumura. Problem 11 which dates from 1845 [2, p.72] is a generalisation of Problem 10 which dates from 1878 [1, p.5], so Problem 10 probably appeared somewhere else at an earlier date.

**10. If $r$ denotes the radius of the small circle in figure 13, show that**

$$1/r = 1/FE + 1/AF.$$

**11. If $r$ and $s$ denote the radii of the two small circles in figure 14, show that**

$$1/\sqrt{rs} = 1/FE + 1/AF.$$

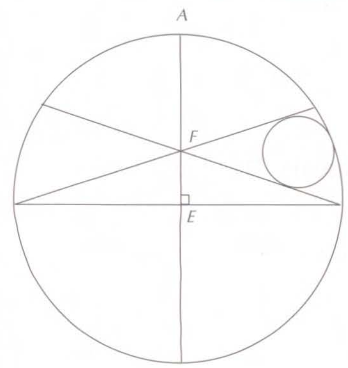


Figure 13

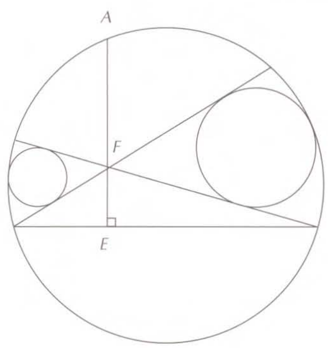


Figure 14

It is not difficult to prove 10 by coordinate geometry, but the calculations involved in investigating 11 are more difficult, and I eventually began to suspect that it is not true. In fact it is possible to show, by considering special cases of figure 14, that the expression $\sqrt{rs}\ (1/FE + 1/AF)$ can take all positive values [6]. Why then should such an incorrect conjecture have been made? We have to make allowances for inaccurate measurements even in a good drawing, and it appears that $F$ has to be very close to the bounding circle before the value of $\sqrt{rs}\ (1/FE + 1/AF)$ differs appreciably from 1. $M^2$

## References


1. H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems*, The Charles Babbage Research Centre, Winnipeg, Canada 1989.
2. Gunma Wasan Study Association, *The Sangaku in Gunma* (in Japanese), 1987.
3. Okayu (ed.), revised Ogawa, *Sampo Semmon Sho*, 1940.
4. H. Okumura, Circle patterns arising from a seven-circle problem, *Crux Mathematicorum* 21 (1995) 213-217.
5. J. F. Rigby, Cirlce problems arising from Wasan, *Symmetry: Culture and Science* (to appear).
6. J. F. Rigby, An incorrect sangaku conjecture, *Symmetry: Culture and Science* (to appear).
7. J. F. Rigby, A sangaku problem, *Mathematics and Informatics Quarterly* (to appear).
8. Sakuma, *Sampo Kigen Shu* Vol 2, 1877.



John Rigby was an undergraduate and postgraduate at Trinity College, Cambridge. He recently retired from his position of Reader in Pure Mathematics at the University of Wales, Cardiff, but is still lecturing on geometry part-time. He has been based in Cardiff since 1959, and has held temporary university appointments in Toronto, Ankara and Singapore. His hobbies include ornamental art, classical music, folk dancing and mountain walking.

Dr. Rigby can be contacted at

*University of Wales, School of Mathematics,*
*Senghennydd Road, P. O. Box No. 926, Cardiff CF2 4YH, UK.*
*e-mail: smajfr@cardiff.ac.uk*