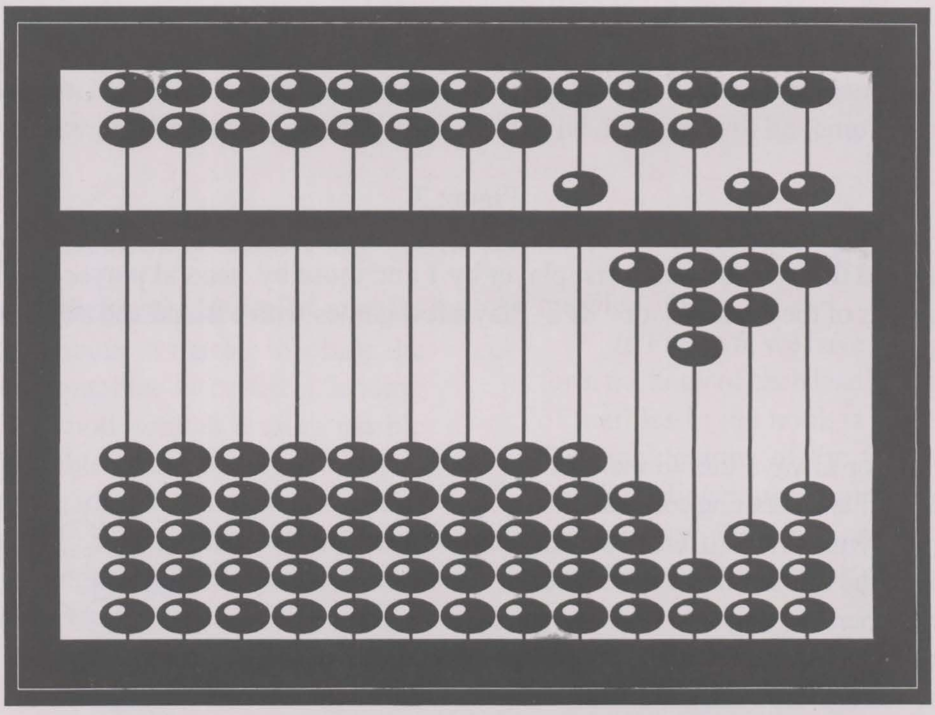
PICK-UP MARBLES AND BEAD-UP ABACUS

Lam Tao Kai and Tan Chong Hui



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1. Pick-up Marbles

This is a simple paper-and-pencil game for two persons. Alternately, it can be played out with actual marbles or ping pong balls. First we need a V-shaped container and some uniformly-sized marbles, just large enough so that when we put them in the container, one marble sits in the first row, two in the second, and so on, and we assume that the marbles all lie in the same plane. Furthermore, marbles occupying the same row do not touch one another. Any configuration is acceptable as long as it is stable — the marbles do not start rolling down by themselves. Let us choose the configuration shown in Figure 1 as our starting configuration.

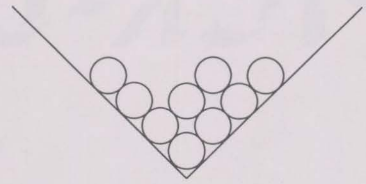


Figure 1

The players alternately remove two adjacent marbles, always ensuring that no marble rolls down after his turn of play. The loser is the one who is not able to remove any marbles on his turn. An example starting with the configuration in Figure 1 is shown below.

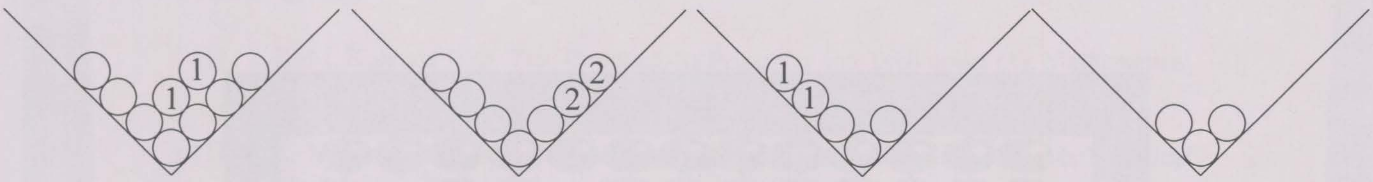


Figure 2

We have marked the moves by the first player by **1** and those by second player by **2**. Note that every move removes marbles of the form or . Play a few games with a friend and see if you can answer the following questions.

1. Is the first player always the winner if the configuration in Figure 1 is the starting configuration? Is there a strategy to win? If a different starting configuration such as the one shown below is used, which player is the winner?
2. At the end of the game, what does the configuration of the marbles look like?

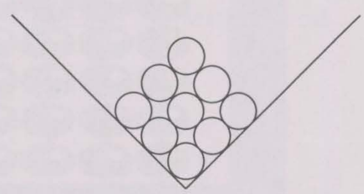
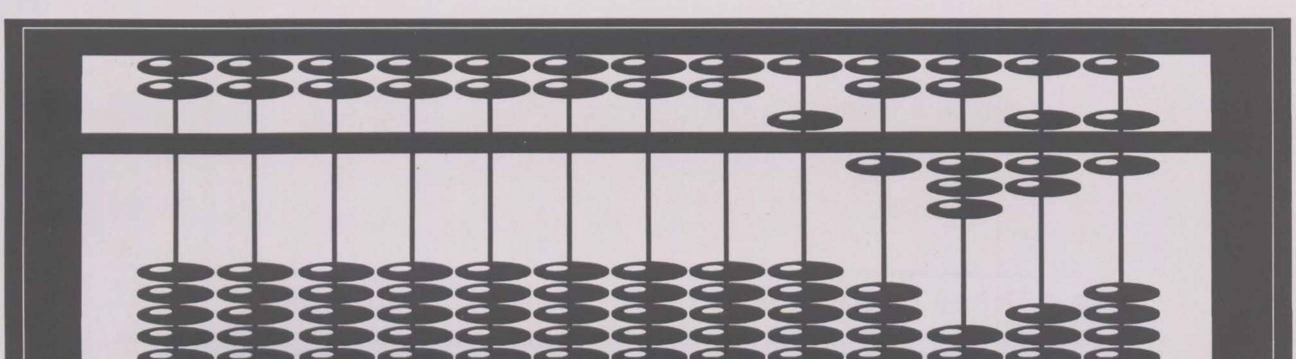


Figure 3



2. Bead-up Abacus

In this section, we describe another two-player game. Suppose we have an abacus with only two rods and some beads on them. The beads can only occupy uniformly spaced positions on the rods. Each player alternately moves a bead up the rod by one position. A bead cannot be moved if either another bead is directly on top of it or it has already hit the top of the abacus. Any player who cannot make a move loses. A sample game is shown below. Again, we have marked the first player's moves by **1** and the second player's moves by **2**. In this game, the first player wins.

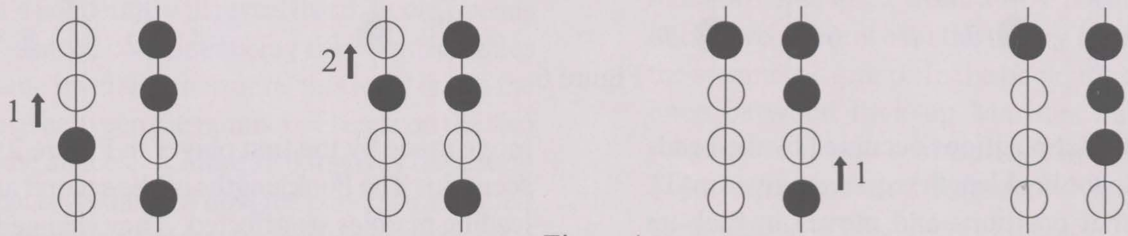


Figure 4

If you play a few games, you will realize that this is easier than Pick-up Marbles. To determine who will win a game, you just have to count the number of moves required to push all the beads to the top. For each bead, we count the number of empty positions above it and then add them all up. If this number is even, then the first player loses; otherwise he wins. For example, in the game illustrated in Figure 4, the left rod holds one bead with 2 empty positions above it. On the right rod, the first two beads are already at the top. The third bead has 1 empty position above it. So the total number of moves is 3, which is odd and thus the first player wins.

Has it ever occurred to you that Pick-up Marbles and Bead-up Abacus are intimately related? In fact, Pick-up Marbles is nothing more than a well-disguised game of Bead-up Abacus. The disguise will be removed in the next section.

3. Pick-up Marbles to Bead-up Abacus

Let us encode the configuration of marbles by a sequence of numbers. A marble touching the right edge of the container is called a leading marble. We call the bottommost leading marble the first leading marble, the leading marble just above it the second leading marble and so on. For each leading marble, we call the collection of marbles that form a right-angled V with the leading marble as the apex the hook of the leading marble.

In Figure 5, we have shaded the marbles that form the hook of each leading marble. The number of marbles in the hook is called the hooklength. The hooklengths of the first, second, third and fourth leading marbles on the right are 7,4,3 and 1 respectively. We write as (7,4,3,1) and call this the hooklength sequence of the configuration.

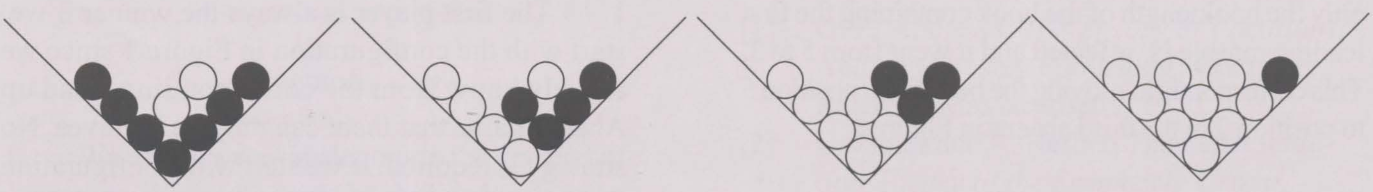


Figure 5

Note that the hooklength sequence is in decreasing order. This is true for any acceptable configuration. Also, different configurations will give rise to different hooklength sequences. As an illustration, the hooklength sequences of all the configurations in the game shown in Figure 2 are

$$(7,4,3,1), (7,3,2,1), (5,1) \quad \text{and} \quad (3,1). \qquad (1)$$

Given an abacus with beads, we will describe a way of labelling the beads. Starting from the top row, we scan the positions on the abacus from left to right, then going to the next row. We label the first empty position by 0 and then every subsequent position, whether empty or occupied, by 1, 2, 3 and so on. We illustrate with the abacus from Figure 4.

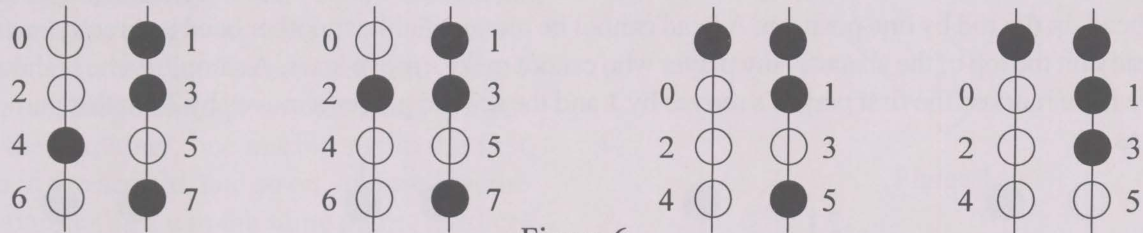


Figure 6

Notice that positions occupied by the beads correspond to the hooklength sequences given in [1]. We claim that positions and moves in Pick-up Marbles can be translated consistently into positions and moves in Bead-up Abacus. Let us describe this in detail.

Given a starting configuration of Pick-up Marbles, say as in Figure 1, we first calculate the hooklength sequence and then arrange the beads in the abacus so that positions corresponding to the hooklengths are occupied. Doing this with the configuration in Figure 1 produces the leftmost abacus arrangement in Figure 6.

For every move in Pick-up Marbles, we make a corresponding move in Bead-up Abacus. This will depend on the relative position of the marbles removed.

Case 1: The two marbles that are removed are in the position .

In this case, only one hooklength is changed. It is decreased by two, say from *a* to a - 2. In the corresponding Bead-up Abacus game, this corresponds to moving the bead in position *a* to position a - 2, that is up one position. For example, in the second move made by the first player in Figure 2, only the hooklength of the hook containing the first leading marble is affected and it went from 5 to 3. This corresponds to moving the bead from position 5 to position 3 in the third abacus in Figure 6.

Case 2: The two marbles that are removed are in the position .

Two kinds of situation can occur. In the first situation, two hooklengths are affected by this move. Furthermore, these hooklengths differ by 1. That is to say, if one of them is *a*, then the other is a - 1. After the move, they become a - 1 and a - 2 respectively. This again corresponds to moving a bead occupying position a up to position a - 2. For example, the first move made by the first player in Figure 2 fits this scenario. The hooklengths of the second and third leading marbles are affected. They changed from 4 and 3 to 3 and 2 respectively. This corresponds to moving the bead from position 4 on the first abacus in Figure 6 to position 2.

The other situation that can occur is when all the hooklengths are affected. For example, the move made by the second player in Figure 2 belongs to this category. In this case, the hooklengths 2 and 1 disappear since the corresponding leading marbles are removed. All other hooklengths are decreased by 2. In Bead-up Abacus, the bead in position 2 on the second abacus is moved to position 0 by the second player. When this occurs, both positions 0 and 1 are occupied. So the first empty position has moved to position 2 and we have to relabel the abacus starting with 0 from this new position. Apart from positions 0 and 1, the old position labels are each reduced by 2 (see Figure 6).

Thus a move in Pick-up Marble corresponds to a move in Bead-up Abacus. The two games are actually the same! With this information, we can answer the questions posed at the end of Section 1.

1. The first player is always the winner if we start with the configuration in Figure 1 since we already know from the corresponding Bead up Abacus game that there can only be 3 moves. No strategy is required. If we start with configuration shown in Figure 3, the corresponding Bead-up Abacus game starts with the beads in the positions shown in Figure 7.

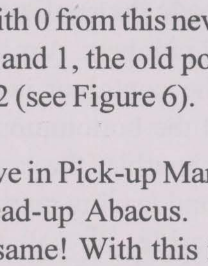


Figure 7

The total number of moves is 4 and so the second player wins.

2. At the end of Bead-up Abacus, all the beads are pushed to the top. When we look at the position labels, we find that the hooklength sequence consists of odd numbers starting from 1. A moment's pondering will reveal that this corresponds to the V-shaped container being filled with marbles to the same level. Furthermore, this level is just the difference between the number of beads on the two rods of the abacus. A more striking way of putting this is that regardless of how the players make their moves in Pick-up Marbles, the final configuration that remains when the game ends is always the same for a fixed initial configuration.

4. Generalization

Pick-up Marbles may be generalized to a game which involves removing q marbles at a time, where q is a fixed number bigger than 2, with an additional rule: the q marbles removed each time must be in one connected piece lying at the top of the V-shaped container. The no-rolling rule still applies.

For example, in Figure 8, the shaded marbles in the configuration on the right are removable while those in the configuration on the left are not. Here $q = 4$.

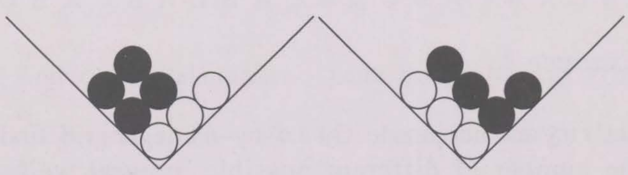


Figure 8

We can, in a way analogous to the first part of this article, mirror the game by shifting beads on an abacus that has q rods. With the increased complexity as q becomes large, it is amazing that the core is unique to the initial configuration and is unaffected by how the q marbles are removed each time. It is also remarkable that the proof is extremely transparent once the game is viewed from a different perspective — via shifting beads up an abacus with q rods. We leave all this to the interested reader.

5. Connections with Other Parts of Mathematics

The two games described in this article may seem to have been invented with recreation as their sole purposes. However, we must remark that our material is adapted from a tiny portion of [2], a definitive reference on the theory of characters of the symmetric group. In the book, the correspondence between Pick-up Marbles and Bead-up Abacus is used to simplify the Murnaghan-Nakayama recursion formula — a rule for the computation of characters of the symmetric group. Characters have been invented to link the theory of groups to the theory of linear algebra, and theorems established for linear algebra can shed light on group theory. An introduction to characters can be found in [3] or [4].

Such inter-relationships are typical of mathematics and they serve to bind its myriad branches into one single unity. Thus, it should not come as a big surprise if from a slightly different perspective, we are able to catch a glimpse of another branch of mathematics in Pick-up marbles. If we rotate the whole setup 135° clockwise so that the apex of the V-shaped container is now in the upper left hand corner, and replace the marbles by squares, we get what is known as a *Ferrers diagram*. (In the French convention, the setup is rotated 45° clockwise so that the apex is in the lower left hand corner). Ferrers diagrams are used to study properties of partitions (see [1]).

From these diagrams we can deduce a bunch of beautiful partition formulas in number theory.

References

[1] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1976.

[2] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Addison-Wesley Publishing Company, Reading, Massachusetts, 1981.

[3] W. Ledermann, *Introduction to Group Characters*, Cambridge University Press, Cambridge, 1977.

[4] F. D. Murnaghan, *The Theory of Group Representations*, The Johns Hopkins Press, Baltimore, 1938; Dover Publications, New York, 1963.