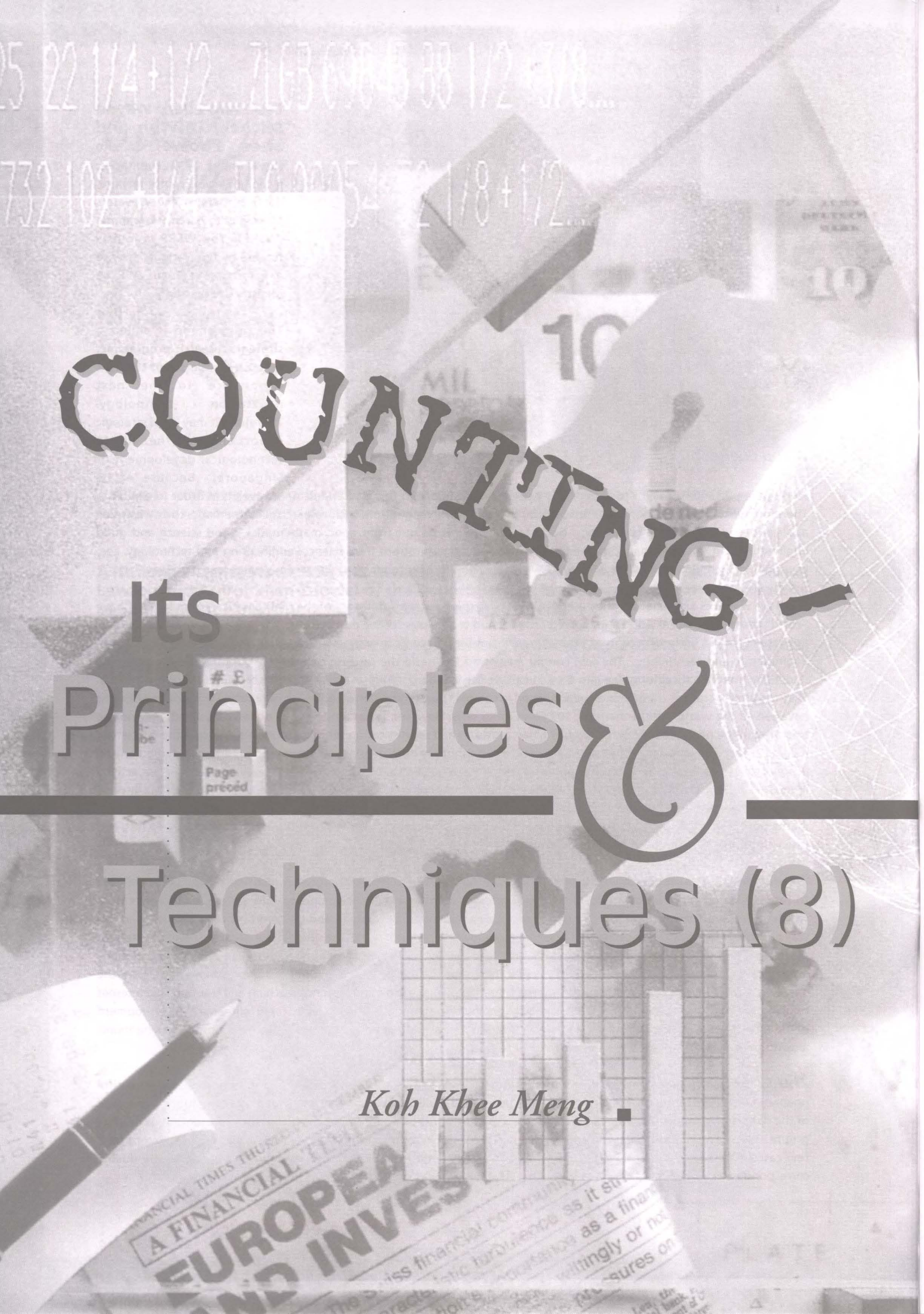

COUNTING – Its Principles & Techniques (8)

Koh Khee Meng

20. General Statement of the Principle of Inclusion and Exclusion

In [7], we introduced the Principle of Inclusion and Exclusion (PIE) by first deriving the identity (see (17.2))

$$|A_1 \cup A_2| = (|A_1| + |A_2|) - |A_1 \cap A_2| \tag{20.1}$$

for two finite sets A_1 and A_2, and then extending it to the following identity (see (18.1)):

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| = &(|A_1|+|A_2|+|A_3|) - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)\\ &+|A_1 \cap A_2 \cap A_3|\end{aligned} \tag{20.2}$$

for three finite sets A_1, A_2 and A_3. Naturally, one would like to know whether (20.1) and (20.2) could be extended to an identity involving any $n(\geq 2)$ finite sets $A_1 A_2, \ldots, A_n$; and if so, what identity one would get in general. The main objective of this article is to deal with this problem. We shall first extend (20.2) to an identity involving four sets, and then by observing these special cases, we will obtain the general statement of the PIE for any finite sets. Finally, two examples showing the application of this general statement will be given in the subsequent sections.

Suppose that four finite sets A_1, A_2, A_3, and A_4 are given. By applying (20.1), (20.2) and some basic laws for sets, we have

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| &= |(A_1 \cup A_2 \cup A_3) \cup A_4|\\
&= |A_1 \cup A_2 \cup A_3| + |A_4| - |(A_1 \cup A_2 \cup A_3) \cap A_4| \quad \text{(by (20.1)}\\
&= |A_1 \cup A_2 \cup A_3| + |A_4| - |(A_1 \cap A_4) \cup (A_2 \cap A_4) \cup (A_3 \cap A_4)|\\
&= (|A_1|+|A_2|+|A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|\\
&\quad + |A_1 \cap A_2 \cap A_3|) + |A_4| - (|A_1 \cap A_4| + |A_2 \cap A_4| + |A_3 \cap A_4|)\\
&\quad - |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4|\\
&\quad + |A_1 \cap A_2 \cap A_3 \cap A_4|. \quad \text{(by (20.2)}\end{aligned}$$

That is,

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = &(|A_1|+|A_2|+|A_3|+|A_4|) - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4|\\
&+ |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)\\
&+ (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|)\\
&- |A_1 \cap A_2 \cap A_3 \cap A_4|. \quad \text{(by (20.3)}\end{aligned}$$

Now, let us look at the identities (20.1) - (20.3) carefully and make some observations on the patterns of the terms on the right-hand sides of the identities.

For the sum of terms within the first grouping, we have:

n	Sum	number of terms in the sum
2	$\|A_1\|+\|A_2\|$	$2=\binom{2}{1}$
3	$\|A_1\|+\|A_2\|+\|A_3\|$	$3=\binom{3}{1}$
4	$\|A_1\|+\|A_2\|+\|A_3\|+\|A_4\|$	$4=\binom{4}{1}$

For the sum of terms within the second grouping, we have:

n	Sum	number of terms in the sum
2	$\|A_1 \cap A_2\|$	$2=\binom{2}{2}$
3	$\|A_1 \cap A_2\|+\|A_1 \cap A_3\|+\|A_2 \cap A_3\|$	$3=\binom{3}{2}$
4	$\|A_1 \cap A_2\|+\|A_1 \cap A_3\|+\|A_1 \cap A_4\|$ $+\|A_2 \cap A_3\|+\|A_2 \cap A_4\|+\|A_3 \cap A_4\|$	$6=\binom{4}{2}$

For the sum of terms within the third grouping, we have:

n	Sum	number of terms in the sum
2	none	0
3	$\|A_1 \cap A_2 \cap A_3\|$	$1=\binom{3}{3}$
4	$\|A_1 \cap A_2 \cap A_3\|+\|A_1 \cap A_2 \cap A_4\|$ $+\|A_1 \cap A_3 \cap A_4\|+\|A_2 \cap A_3 \cap A_4\|$	$4=\binom{4}{3}$

We also notice that the groupings alternate in sign, beginning with a (+) sign.

Suppose now we are given n finite sets: A_1, $A_2, \ldots, A_n$. By generalizing the above observations, what identity do you expect for $|A_1 \cup A_2 \cup \cdots \cup A_n|$?

The first grouping should be the sum of $\binom{n}{1} = n$ terms involving single set:

$$|A_1|+|A_2|+\cdots+|A_n|;$$

in abbreviation,

$$\sum_{i=1}^{n}|A_i|.$$

The second grouping should be the sum of $\binom{n}{2}$ terms involving intersection of two sets:

$$|A_1 \cap A_2|+|A_1 \cap A_3|+\cdots+|A_{n-1} \cap A_n|;$$

in abbreviation,

$$\sum_{i<j}|A_i \cap A_j|.$$

The third grouping should be the sum of $\binom{n}{3}$ terms involving intersection of three sets:

$$|A_1 \cap A_2 \cap A_3|+|A_1 \cap A_2 \cap A_4|+\cdots+|A_{n-2} \cap A_{n-1} \cap A_n|;$$

in abbreviation,

$$\sum_{i<j<k}|A_i \cap A_j \cap A_k|.$$

Likewise, the forth grouping should be the sum of $\binom{n}{4}$ terms involving intersection of four sets;

$$\sum_{i<j<k<l}|A_i \cap A_j \cap A_k \cap A_l|;$$

and so on.

Bearing in mind that the groupings alternate in sign, beginning with a (+) sign, one would expect that the following holds:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|. \quad (20.4)$$

Indeed, it can be proved (for instance, by mathematical induction) that (20.4) holds for any n finite sets $A_1, A_2, \ldots, A_n$; and (20.4) is regarded as the general statement for the PIE.

21. Arrangements in a Row

In this section, we shall show the first application of (20.4) by considering the following:

Example 21.1 How many ways are there to arrange $n(\geq 2)$ married couples in a row so that at least one couple are next to each other?

Denote the n couples by $H_1, W_1, H_2, W_2, \ldots, H_n, W_n$. Thus, when $n = 4$ for example, the following arrangements are possible:

$$W_2H_1H_4\underline{W_3H_3}H_2W_1W_4, \quad H_4\underline{H_2W_2}W_4H_3\underline{W_1H_1}H_4.$$

Solving the above problem by dividing it into cases such as exactly one couple are next to each other, exactly two couples are next to each other, and so on would be very complicated. Let us try to apply (20.4).

For each $i = 1,2,\ldots,n$, let A_i be the set of arrangements of the n couples such that H_i and W_i and are adjacent (next to each other). The problem is thus to enumerate $|A_1 \cup A_2 \cup \ldots \cup A_n|$.

To apply (20.4), we compute each grouping on its right-hand side.

To compute $\sum_{i=1}^{n} |A_i|$, we first consider $|A_1|$. A_1 is the set of arrangements of the n couples such that H_1 and W_1 are adjacent. This is same as arranging the $2n-1$ objects:

$$H_1W_1,\ H_2, W_2, \ldots, H_n, W_n$$

in a row where H_1W_1 can be permuted in two ways: H_1W_1 and W_1H_1. Thus

$$|A_1| = 2 \cdot (2n-1)!$$

Similarly, $|A_i| = 2 \cdot (2n-1)!$ For each $i = 2,\ldots,n$. Thus

$$\sum_{i=1}^{n} |A_i| = \binom{n}{1} \cdot 2 \cdot (2n-1)!$$

To compute $\sum_{i<j} |A_i \cap A_j|$, we first consider $|A_1 \cap A_2|$. $A_1 \cap A_2$ is the set of arrangements of the n couples such that H_1 and W_1 are adjacent and H_2 and W_2 are adjacent. This is same as arranging the $2n-2$ objects:

$$H_1W_1, H_2W_2, H_3, W_3, \ldots, H_n, W_n$$

in a row where both H_1W_1 and H_2W_2 can be permuted by themselves. Thus

$$|A_1 \cap A_2| = 2^2 \cdot (2n-2)!$$

Similarly, for $1 \leq i < j \leq n$

$$|A_i \cap A_j| = 2^2 \cdot (2n-2)!$$

Thus

$$\sum_{i<j} |A_i \cap A_j| = \binom{n}{2} 2^2 \cdot (2n-2)!$$

We now leave it to the reader to show that

$$\sum_{i<j<k} |A_i \cap A_j \cap A_k| = \binom{n}{3} 2^3 \cdot (2n-3)!,$$

and so on to obtain the following final result that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} 2^r \cdot (2n-r)!$$

For the case when $n = 4$, we have

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \binom{4}{1} 2 \cdot (8-1)! - \binom{4}{2} \cdot 2^2 \cdot (8-2)! + \binom{4}{3} \cdot 2^3 \cdot (8-3)! - \binom{4}{4} \cdot 2^4 \cdot (8-4)!$$
$$= 8 \cdot 7! - 24 \cdot 6! + 32 \cdot 5! - 16 \cdot 4!$$
$$= 26496.$$

22. Derangements

In this section, we shall introduce an old problem on deck of cards. Two decks X, Y of cards, with 52 cards each, are given. The 52 cards of X are first laid out. Those of Y are then placed randomly with one at the top of a card of X so that 52 pairs of cards are formed. The question is: what is the probability that no cards in each pair are identical (i.e., having the same suit and rank)? This problem, known as "le problÌme des rencontres" (the matching problem), was introduced and studied by the Frenchman Pierre RÈmond de Montmort (1678-1719) around 1708. The number of ways of distributing the cards of Y to form 52 pairs of cards with those in X is clearly 52! Thus, to find the desired probability, we need to find out the number of ways of distributing the cards of Y such that each card in Y is placed at the top of a different card in X.

Instead of solving the above problem directly, let us generalize it and consider the following more general problem For each positive integer n, let $N_n = \{1,2,\ldots,n\}$. A permutation $a_1a_2\ldots a_n$ of N_n (see Section 4 [2]) is called a *derangement* of N_n if $a_i \neq i$ for each $i = 1,2,\ldots,n$. Thus 54132 is a derangement of N_5 but 51<u>342</u> and 3<u>2</u>154 are not. For $n = 1,2,3,4$, all the derangements of N_n are shown in the following table.

n	derangements
1	none
2	21
3	231, 312
4	2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

Let D_n denoe the number of derangements of N_n. It follows from the above table that $D_1 = 0$, $D_2 = 1$, $D_3 = 2$ and $D_4 = 9$. Returning back to the matching problem, it is now clear that its answer is given by $\frac{D_{52}}{52!}$. How to evalute D_n for each n? After some thought you may realize that this is not a trivial problem. Well, we are given a good opportunity to show our second application of PIE.

Before we proceed any further, let us first derive an equivalent form of (20.4).

For a subset A of a univeral set S, let $\overline{A}$ denote its *complement*. It was pointed out in [7] that (20.2) is equivalent to the following: For any subsets A_1, A_2, A_3 of S,

$$|\overline{A_1}\cap\overline{A_2}\cap\overline{A_3}|=|S|-(|A_1|+|A_2|+|A_3|)+(|A_1\cap A_2|+|A_1\cap A_3|+|A_2\cap A_3|)-|A_1\cap A_2\cap A_3|.$$

In general, for any $n(\geq 2)$ subsets $A_1, A_2,\ldots, A_n$ of S, one can show that (20.4) is equivalent to the following:

$$\begin{aligned}|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|&=|S|-|A_1\cup A_2\cup\cdots\cup A_n|\\&=|S|-\sum_{i=1}^{n}|A_i|+\sum_{i<j}|A_i\cap A_j|-\sum_{i<j<k}|A_i\cap A_j\cap A_k|\\&\quad+\cdots+(-1)^n|A_1\cap A_2\cap\cdots\cap A_n|.\end{aligned}\qquad(22.1)$$

We shall now evaluate D_n by applying (22.1). Let us first identify what the universal set is. We are now concerned with derangements, which are special types of permutations of N_n. So, let the universal set S be the set of all permuations of N_n.

For each $i = 1,2,\ldots,n$, let A_i be the set of permutations $a_1a_2\ldots a_n$ in S such that $a_i = i$. Thus $\overline{A_i}$ is the set of permutations in S such that $a_i \neq i$ and so $\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}$ is the set of permutations in S such that $a_i\neq i$ for all $i = 1,2,\ldots,n$, which is exactly the set of derangements of N_n. We thus have

$$D_n=|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|.$$

To evaluate D_n by (22.1), we evalute each grouping on the right-hand side of (22.1). Clearly, as S is the set of all permutations of N_n, we have $|S|=n!$.

Observe that A_1 is the set of permutations of the form $1a_2a_3\ldots a_n$. Thus Similarly, $|A_i|=(n-1)!$ for each $i = 2,\ldots,n$, and so

$$\sum_{i=1}^{n}|A_i|=n\cdot(n-1)!=\binom{n}{1}\cdot(n-1)!$$

As $A_1\cap A_2$ is the set of permutations of the form $12a_3\ldots a_n$, we have $|A_1\cap A_2|=(n-2)!$ Similarly, $|A_i\cap A_j|=(n-2)!$ for all $i, j \in \{1,2,\ldots,n\}$ with $i < j$, and so

$$\sum_{i<j}|A_i\cap A_j|=(n-2)!$$

We now leave it to the reader to show that

$$\sum_{i<j<k}|A_i\cap A_j\cap A_k|=\binom{n}{3}\cdot(n-3)!$$

and so on to obtain the following final result by (22.1) that

$$\begin{aligned}D_n&=|\overline{A_1}\cap\overline{A_2}\cap\cdots\cap\overline{A_n}|\\&=n!-\binom{n}{1}\cdot(n-1)!+\binom{n}{2}\cdot(n-2)!-\binom{n}{3}\cdot(n-3)!\\&\quad+\cdots+(-1)^n\binom{n}{n}\cdot(n-n)!\end{aligned}$$

Note that for $r = 1,2,\ldots,n$,

$$\binom{n}{r}\cdot(n-r)!=\frac{n!}{r!(n-r)!}(n-r)!=\frac{n!}{r!}.$$

Thus,

$$\begin{aligned}D_n&=n!-\frac{n!}{1!}+\frac{n!}{2!}-\frac{n!}{3!}+\cdots+(-1)^n\frac{n!}{n!}\\&=n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right).\end{aligned}$$

Suppose we generate a permutation of N_n at random. The probability that this permutation is a derangement is given by $\frac{D_n}{n!}$ which by the above result, is

$$\frac{D_n}{n!}=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}.$$

When n gets larger and larger, it is know that the quotient $\frac{D_n}{n!}$ gets closer and closer to $\frac{1}{e}(\approx 0.367)$, where the constant e, called the natural exponential base, is defined by $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. It is known that $e \approx 2.718281828459045$. (The letter 'e' was chosen in honor of the great Swiss mathematician L. Euler (1707-1783) who made some significant contributions to the study of problems related to the above limit.)

Problem 22.1 Show that the number of integer solutions to the equation (see Section 8 [3])

$$x_1+x_2+\cdots+x_{11}=100$$

such that $0\leq x_r\leq 9$ for each $r = 1,2,\ldots,11$ is given by

$$\sum_{r=0}^{11}(-1)^r\binom{11}{r}\binom{10(11-r)}{10}.$$

Problem 22.2 Each of ten ladies checks her hat and umbrella in a cloakroom and the attendent gives each lady back a hat and an umbrella at random. Show that the number of ways this can be done so that no lady gets back both of her possessions is

$$\sum_{r=0}^{10}(-1)^r\binom{10}{r}\{(10-r)!\}^2.$$

Problem 22.3 Show that the number of onto mappings (see Section 19 [7]) from N_m to N_n, where $m\geq n\geq 1$, is given by

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)^m.$$

Probelm 22.4 For $r = 1,2,\ldots,2000$, let A_r be a set such that $|A_r| = 44$. Assume that $|A_i\cap A_j| = 1$ for all $i, j \in \{1, 2,\ldots, 2000\}$ with $i\neq j$. Evaluate

$$\left|\bigcup_{r=1}^{2000}A_r\right|.$$

(Ans. 86001)

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