Contest

1. Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be typed and sent to : The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore117543 ; and should arrive before 15 March 2002.

4. The Editor's decision will be final and no correspondence will be entertained.

Problems Corner

Problem 1.

1. (Proposed by Dr Roger Poh, NUS) Prove that every positive rational number can be expressed as a finite series in the form of

$$\frac{1}{p_1} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_2 p_3} + \cdots + \frac{1}{p_1 p_2 \cdots p_k},$$

where $k, p_1, p_2, \ldots, p_k$ are positive integers with $p_1 \le p_2 \le \ldots \le p_k$.

(One $150 book voucher)

Problem 1.

2. Find all positive prime integers p such that for each positive prime integer $q < p$, $p - \lfloor p/q \rfloor q$ contains no square factors.

(One $150 book voucher)

SOLUTIONS

Solutions to the problems of volume 28, No.1, 2001

Problem 1

The numbers 1, 2, 3, ..., 2001 are arranged in a sequence. If the first term is k, then the first k terms of this sequence is rearranged in the reversed order. Is it always possible to obtain 1 as the first term by applying a finite number of this operation to the sequence?

Prize: One $150 book Voucher

Solution I

First it is clear that we will always end up with a repetitive set of sequences no matter what sequence we start with as there are only finitely many permutations of the 2001 numbers. Let the largest number amongst the first terms of all the sequences in this repetitive set be N. If $N \neq 1$, then there are at least two sequences in this set. Thus the following 2 sequences must exist:

$N, X_1, X_2, \ldots, X_{2000}$, where N is the first term,
$X_{N-1}, X_{N-2}, \ldots, X_1, N, X_{N+1}, \ldots, X_{2000}$, where N is the Nth term.

with the first sequence leading to the second upon undergoing the operation.

However, it is impossible for the second sequence to permute back to the first one as all $X_1, X_2, \ldots, X_{N-1}$ are less than N. This contradiction shows that N must be 1.

by **Ong Xing Cong**
Raffles Institution

Solution II

We shall use induction to show that for any given permutation of $1, 2, \ldots, n$, it is always possible to obtain 1 as the first term by applying a finite number of the operation. This is certainly true when $n = 1$ and 2. Suppose that this is true for any permutation of $1, 2, \ldots, n$ with $n \geq 2$. Consider a permutation of $1, 2, \ldots, (n+1)$. Let this permutation be $a_1, a_2, \ldots, (n+1), \ldots, a_n$. The number $(n+1)$ will fall in three possible places: (i) in the front, (ii) at the back or (iii) somewhere in the middle. If $(n+1)$ is the first term, then by applying one operation to the sequence, it will bring $(n+1)$ to the last term. That is, case (i) can be reduced to case (ii). For case (ii), induction hypothesis applied to the first n terms of the sequence implies that the sequence can be reduced to one having 1 as the first term. For case (iii), we may treat $(n+1)$ as a_n and either apply induction hypothesis to conclude that it is possible to bring 1 to the first position by applying a finite number of the operation to the first n terms of this sequence or it eventually gives a sequence having $(n+1)$ as the first term, thus reducing to case (ii).

by **Soh Yong Sheng**
Raffles Institution

Editor's note: Solved also by Joel Tay Wei En (Anglo Chinese School Independent), Gideon Tan Guanyuan (Raffles Institution), Gary Yen Yuanlong (Angle Chinese Junior College), Wee Hoe Teck (Massachusetts Institute of Technology), Dinh Thi Thi, (Raffles Girls' Secondary School). The prize was shared equally between by Ong Xing Cong and Soh Yong Sheng.

SOLUTIONS

Solutions to the problems of volume 28, No.1, 2001

Problem 2

Let b and c be positive integers such that b divides $c^2 + 1$ and c divides $b^2 + 1$. Determine the value of

$$\frac{b}{c}+\frac{c}{b}+\frac{1}{bc}.$$

• Prize

One $150 book Voucher

Solution I

First b and c must be relatively prime so that the condition $b|(c^2+1)$ and $c|(b^2+1)$ is equivalent to $bc|(b^2+c^2+1)$. Now we prove that $bc|(b^2+c^2+1)$ implies that $b^2+c^2+1=3bc$.

Since bc divides b^2+c^2+1, we have c divides $b+(c^2+1)/b$. Hence, c divides $(b+(c^2+1)/b)^2-(b^2+c^2+1)-c^2=c^2+((c^2+1)/b)^2+1$. Also $(c^2+1)/b$ divides $c^2+((c^2+1)/b)^2+1$. As c and $(c^2+1)/b$ are relatively prime, we have $c(c^2+1)/b$ divides $c^2+((c^2+1)/b)^2+1$. Similarly, $b(b^2+1)/c$ divides $b^2+((b^2+1)/c)^2+1$.

If $b>c$, then $bc\geq c^2+1$. Let $d=(c^2+1)/b$. Then $c\geq(c^2+1)/b=d$. If $c>b$, then by letting $d=(b^2+1)/c$ we have $b\geq d$. This shows that from any (b,c) that satisfy the conditions $b\neq c$ and $bc|b^2+c^2+1$, we get the pair (x,y) with $\min\{b,c\}=\max\{x,y\}$ such that $xy|x^2+y^2+1$. If $x\neq y$, we can repeat the process. But the pairs of positive integers so obtained cannot be infinitely small. At certain stage, we must have $x=y$ at which the condition $xy|x^2+y^2+1$ implies that $x=y=1$. Also with $x=y=1$, we have $x^2+y^2+1=3xy$. Lastly, it can be easily verified that if $c^2+d^2+1=3cd$ with $d=(c^2+1)/b$ or $b^2+d^2+1=3bd$ with $d=(b^2+1)/c$, then $b^2+c^2+1=3bc$. Consequently, for the original pair (b,c), we have $b^2+c^2+1=3bc$. That is the value of $\frac{b}{c}+\frac{c}{b}+\frac{1}{bc}$ is 3.

by **Joel Tay Wei En**
Anglo Chinese School Independent

SOLUTIONS

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Solution II

If (x, y) is a pair of positive integers and satisfies x divides y^2+1 and y divides x^2+1 with $x \geq y$, we shall call the pair (x, y) good. Without loss of generality, assume (b, c) is good, If $b = c$, then c divides c^2+1 implies that $b = c = 1$. If $b > 1$, $b > c$, let $c^2 + 1 = bx$. Thus $c \geq x$, and equality holds iff $c = x = 1$. We have $b^2 = [(c^2+1)/x]^2 \equiv -1 \pmod c$. Thus $1/x^2 \equiv -1 \pmod c$ so that $x^2 \equiv -1 \pmod c$. Also we have $c^2 \equiv -1 \pmod x$. Thus, (c, x) is good. Therefore, for any good pair (b, c) with $b > 1$, it is always possible to find another good pair $(c, (c^2+1)/b)$. As there are only finitely many integers less than b, a descending sequence starting with the good pair (b, c) will always end up with a pair of two equal numbers which must be $(1, 1)$. Consequently, all good pairs come from the sequence: $(1,1), (2,1), (5,2), (13,5), (34,13), \ldots (p_n, q_n) \cdots$. That is the sequence defined by: $(p_1, q_1) = (1, 1)$ and for $n \geq 1$, $p_n = (p_{n-1}^2 + 1)/q_{n-1}$ and $q_n = p_{n-1}$. We also have

$$\frac{p_{n-1}}{q_{n-1}} + \frac{q_{n-1}}{p_{n-1}} + \frac{1}{p_{n-1}q_{n-1}} = \frac{p_n}{q_n} + \frac{q_n}{p_n} + \frac{1}{p_n q_n}.$$

Thus, the value of $\frac{p_n}{q_n} + \frac{q_n}{p_n} + \frac{1}{p_n q_n}$ is constant for all $n \geq 1$ in the sequence. When $n = 1$, $\frac{p_n}{q_n} + \frac{q_n}{p_n} + \frac{1}{p_n q_n} = 3$. Thus the value needed is 3.

by **Gideon Tan Guanyuan**
Raffles Institution

Editor's note: There are several ways to get the descent going. One may use induction on $b+c$ or another way suggested by Wee Hoe Teck is as follow:
Take any $k \in \mathbb{Z}^+, k \neq 3$, and let

$$S_k = \{(a, b) \mid a \mid b^2+1, b \mid a^2+1, \frac{a}{b} + \frac{b}{a} + \frac{1}{ab} = k\}.$$

Suppose on the contrary that $S_k \neq \emptyset$. Then, choose $(a_0, b_0) \in S_k$ such that $\max(a_0, b_0)$ is minimum over all pairs (a, b) in S_k. Without loss of generality, we may assume $a_0 \geq b_0$, and thus $a_0 > b_0$ since $k \neq 3$. Observe that a_0 is a root of the quadratic equation $t^2 - kb_0t + b_0^2 + 1 = 0$, and that the other root is given by $\frac{b_0^2+1}{a_0} \in \mathbb{Z}^+$. Hence, $(b_0, \frac{b_0^2+1}{a_0}) \in S_k$. Moreover, $b_0^2 < a_0b_0$, so $b_0^2 + 1 \leq a_0b_0$ and $\frac{b_0^2+1}{a_0} \leq b_0 < a_0$, and thus $\max\left(b_0, \frac{b_0^2+1}{a_0}\right) < a_0 = \max(a_0, b_0)$, which contradicts our choice of (a_0, b_0). Hence, $S_k = \emptyset$ for all $k \neq 3$, and the claim follows.

Solved also by Gary Yen Yuanlong (Angle Chinese Junior College), Wee Hoe Teck (Massachusetts Institute of Technology), Dinh Thi Thi, (Raffles Girls' Secondary School). The prize was shared equally between by Joel Tay Wei En and Gideon Tan Guanyuan.