An Elementary Proof of the Non-existence of Certain Polygonal Repunits

Chen Zhuo, Lu Xiaonan, Yan Tianqi

Abstract

In this paper, we demonstrate that 1 is the only integer that is both a repunit and a s-polygonal number, whenever s is a multiple of 4 or 5.

A *repunit* is any number of the form of $1, 11, 111, 1111, \ldots$, i.e. a number that contains only the digit 1 in base 10. Thus a repunit can be represented as

$$R(k) = \frac{10^k - 1}{10 - 1} = \frac{10^k - 1}{9}.$$

A *polygonal number* is a number that can be represented as dots arranged in the shape of a regular polygon. The general expression for a polygonal number with s sides and of rank n [2, p. 190] is

$$P_s(n) = \frac{1}{2}n\big((s-2)n - (s-4)\big).$$

In a recent article, Jaroma [3] used elementary methods to show that the number 1 is the only repunit that is also a triangular number ($s = 3$). Motivated by this, we shall call a positive integer that is both a repunit and a polygonal number with s sides, a *s-polygonal repunit*. It is easy to see that there are two trivial cases of s-polygonal repunits. For any s, $P_s(1) = 1$ is always a s-polygonal repunit. Secondly, since $P_s(2) = s$, we will get a s-polygonal repunit if s is itself a repunit.

Jaroma's result means that there are no non-trivial triangular repunits. Given that repunits and polygonal numbers have fascinated mathematicians and amateurs alike, – for example Beiler [2] devoted a chapter in his book to each topic – it is somewhat surprising to see such a result appearing in print ten years into the 21th century. A detailed literature search revealed that Jaroma was not the first to consider the problem. Ballew and Weger [1] had in fact proved the stronger result that $1, 3, 6, 55, 66$ and 666 are the only triangular numbers that are repdigits – numbers that contain only one repeated digit in base 10. Keith [4, 5] subsequently studied repdigits that are simultaneously s-polygonal numbers via an algorithmic approach. He concluded that there are exactly 1526 distinct polygonal repdigits with 50 or fewer digits.

In this note, we extend the method of Jaroma to prove that besides the number 1, there are no other s-polygonal repunits, whenever s is a multiple of 4 or 5. This result is in a sense the best possible since there exists non-trivial repunits of sides $s = 9, s = 22, s = 38$ and $s = 131$, specifically

$$P_9(6) = P_{38}(3) = 111, P_{22}(11) = 1111, \text{ and } P_{131}(42) = 111111.$$

We are now ready to prove our first theorem.

Theorem 1. *If* $s \equiv 0 \pmod 5$, *then there are no other* s*-polygonal repunits beside* 1.

Proof. Let $m = s - 2 \equiv 3 \pmod 5$ and $k > 1$. To find non-trivial s-polygonal repunits, we equate

$$\frac{10^k - 1}{9} = \frac{mn^2 - (m-2)n}{2}. \tag{1}$$

Clearing denominators and subtracting $20n$ from both sides give

$$\begin{aligned}(9mn-2)(n-1) &= 2 \times 10^k - 20n \\ &= 5(4 \times 10^{k-1} - 4n).\end{aligned} \tag{2}$$

Since $m \equiv 3 \pmod 5$, we have

$$(9mn-2)(n-1) \equiv (27n-2)(n-1) \equiv 2(n-1)^2 \equiv 0 \pmod 5.$$

So $n = 5t + 1$ for some t. But this is impossible for if we substitute such a value of n, the two sides of (2) would have different remainders modulo 25. Hence we are forced to conclude that (1) has no integer solutions under the given conditions. □

Theorem 2. *If* $s \equiv 0 \pmod 4$, *then there are no other* s*-polygonal repunits beside* 1.

Proof. Let $m = s - 2 \equiv 2 \pmod 4$ and $k > 1$. Clearing denominators and subtracting 18 from (1) gives

$$20(10^{k-1} - 1) = (3mn + 6)(3n - 3). \tag{3}$$

As $m \equiv 2 \pmod 4$, we have

$$(3mn+6)(3n-3) \equiv (6n+6)(3n-3) \equiv 18(n+1)(n-1) \equiv 0 \pmod 4.$$

This means that n must be odd. But if n is odd, the left hand side of (3) would be exactly divisible by 4 while the right hand side is at least divisible by 8. Thus no integer solutions exist. □

Acknowledgement

We would like to express gratitude to Professor Toh Pee Choon of National Institute of Education for his invaluable and patient guidance; and Mr Yee Weng Hong of Hwa Chong Institution, for his unwavering support in different ways.

References

[1] D. W. Ballew and R. C. Weger, *Repdigit Triangular Numbers,* J. Recreational Math., **8** (1975/76), 96–98.
[2] A. H. Beiler, *Recreations in the theory of numbers,* 2nd edition, Dover Publications, New York, 1966.
[3] J. H. Jaroma, *Triangular repunit – there is but 1,* Czechoslovak Math. J., **60** (2010), 1075–1077.
[4] M. Keith, *Repdigit polygonal numbers,* J. Recreational Math., **12** (1979/80), 9-15.
[5] M. Keith, *On Repdigit polygonal numbers,* J. Integer Seq., **1** (1998), Article 98.1.6. http://www.cs.uwaterloo.ca/journals/JIS/keith.html

The authors are former students of Hwa Chong Institution (JC).