How to Derive Formulas for Sums of Consecutive Powers

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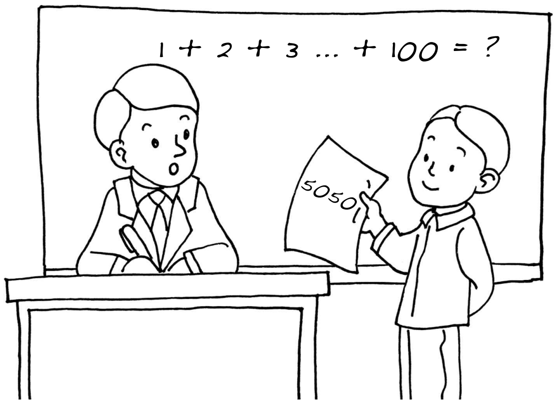


1. A generalization of the formula for consecutive integers

The story of young C.F. Gauss on his ability to sum consecutive integers from 1 to 100 rapidly is well known. Gauss' teacher had wanted to keep his class busy and asked the students to add all integers from 1 to 100. Before the teacher could do some work while his class work on the problem, Gauss gave his teacher the answer 5050. The formula Gauss used for summing the integers is likely to be

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \tag{1}$$

Note that when $n = 100$, the number on the right hand side is 5050.

One question that one can ask is : "Would Gauss' teacher have more time if he had asked his students to sum $1^2, 2^2, 3^2$ all the way up to 100^2?" The answer would be "no" if Gauss knew the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2}$$

Formula (2) is usually introduced in secondary school (or high school) when students first encounter the concept of mathematical induction. Even though most students can prove (2) using mathematical induction, few students are able to derive (2) and often need to memorize the formula by heart.

In this short article, we will prove the following generalization of (1):

Theorem 1.1. *Let a be a positive integer and m be a non-negative integer. Define*

$$(a)_0 = 1$$

and

$$(a)_m = a(a+1)(a+2)\cdots(a+m-1), m \in \mathbf{Z}^+.$$

Then

$$\sum_{k=1}^{n} \frac{(k)_m}{m!} = \frac{(n)_{m+1}}{(m+1)!}. \tag{3}$$

The expression $(a)_m$ is sometimes called the rising factorial. In fact, for integer $m \geq 1$, $(1)_m = m!$.

Observe that if we set $m = 1$ in Theorem 1.1, we recover (1). Theorem 1.1 is different from the usual generalization of (1), which is a formula for sum of consecutive m powers where m is a positive integer. Note that one can retrieve the sum of consecutive m powers from Theorem 1.1 and vice versa but Theorem 1.1 is easier to remember.

Theorem 1.1 is not new as two special cases corresponding to $m = 2$ and 3 appeared in [2, p.6, Problem 1.1.1(c)] and [2, p.384, Problem 1(a)], respectively. They are stated as problems that require the use of mathematical induction.

We now give three proofs of Theorem 1.1.

Proof.

First proof (Induction on n)

The first proof is similar to the solutions to the problems in [2]. We use induction on n. Let m be any positive integer. When n is 1, both sides of (3) are 1.

Suppose (3) is true for $n = \ell - 1$, in other words,

$$\sum_{k=1}^{\ell-1} \frac{(k)_m}{m!} = \frac{(\ell-1)_{m+1}}{(m+1)!}.$$

Then for $n = \ell$, we find that

$$\begin{aligned}\sum_{k=1}^{\ell} \frac{(k)_m}{m!} &= \frac{(\ell)_m}{m!} + \sum_{k=1}^{\ell-1} \frac{(k)_m}{m!} = \frac{(\ell)_m}{m!} + \frac{(\ell-1)_{m+1}}{(m+1)!} \\ &= \frac{(\ell)_m}{(m+1)!}[(m+1) + (\ell-1)] = \frac{(\ell)_{m+1}}{(m+1)!}.\end{aligned}$$

Second proof (Induction on m)

For the second proof, we use induction on m. Although this proof is more complicate than the first proof, the aim is to provide an example of obtaining results by changing the order of summation.

The case for $m = 0$ is immediate.

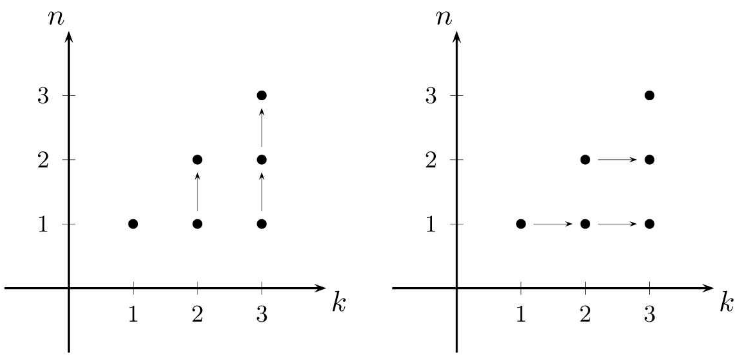
Next assume the result holds for $m = \ell - 1$. Then we have

$$\sum_{k=1}^{n} \frac{(k)_{\ell-1}}{(\ell-1)!} = \frac{(n)_\ell}{\ell!}. \tag{4}$$

Summing both sides from $n = 1$ to N, we find that

$$\sum_{n=1}^{N} \sum_{k=1}^{n} \frac{(k)_{\ell-1}}{(\ell-1)!} = \sum_{n=1}^{N} \frac{(n)_\ell}{\ell!}. \tag{5}$$

Observe that the double sum on the left hand side of (5) can be interchanged so that we evaluate the sum over n first as illustrated in the figure below.



Consequently, we deduce that

$$\begin{aligned}
\sum_{n=1}^{N}\sum_{k=1}^{n}\frac{(k)_{\ell-1}}{(\ell-1)!} &= \sum_{k=1}^{N}\frac{(k)_{\ell-1}}{(\ell-1)!}\sum_{n=k}^{N}1 \\
&= \sum_{k=1}^{N}\frac{(k)_{\ell-1}}{(\ell-1)!}(N-k+1) \\
&= \sum_{k=1}^{N}\frac{(k)_{\ell-1}}{(\ell-1)!}(N+\ell-k-\ell+1) \\
&= (N+\ell)\frac{(N)_\ell}{\ell!} - \ell\sum_{k=1}^{N}\frac{(k)_\ell}{\ell!},
\end{aligned}$$

where we have used (4). Combining the last expression with the right hand side of (5), we find that

$$\sum_{k=1}^{N}\frac{(k)_\ell}{\ell!} = \frac{1}{\ell+1}(N+\ell)\frac{(N)_\ell}{\ell!} = \frac{(N)_{\ell+1}}{(\ell+1)!}$$

and this completes the proof.

Third proof (Use of generating functions)

Our third proof involves showing that both sides of (3) are coefficients of the same power series. As a simple exercise in calculus, one can show that

$$\sum_{\ell=0}^{\infty}\frac{(n)_\ell}{\ell!}z^\ell = \frac{1}{(1-z)^n}. \tag{6}$$

By rearranging the order of summation and applying (6), we observe that

$$\sum_{\ell=0}^{\infty}z^\ell\sum_{n=1}^{N}\frac{(n)_\ell}{\ell!} = \sum_{n=1}^{N}\frac{1}{(1-z)^n} = \frac{1}{z}\left(\frac{1}{(1-z)^N}-1\right).$$

On the other hand, from (6), we see that

$$\sum_{\ell=0}^{\infty}\frac{(N)_{\ell+1}}{(\ell+1)!}z^\ell = \frac{1}{z}\left(\frac{1}{(1-z)^N}-1\right).$$

These computations imply (3).

Fourth proof (Telescopic sum)

The last proof uses the method of telescopic sum. We first observe that

$$\begin{aligned} &k(k+1)(k+2)\cdots(k+m)-(k-1)(k)(k+1)\cdots(k+m-1)\\ &\quad=k(k+1)\cdots(k+m-1)(m+1). \end{aligned}$$

Summing both sides from $k=1$ to $k=n$, we find that the left hand side "telescopes" to $n(n+1)(n+2)\cdots(n+m)$ and we conclude that

$$\sum_{k=1}^{n}k(k+1)\cdots(k+m-1)(m+1)=n(n+1)(n+2)\cdots(n+m).$$

Dividing both sides by $(m+1)!$, we conclude the proof of (3).

□

Remark 1.2. Identity (3) should be compared to Faulhaber's formula [3, p.107]

$$\sum_{k=1}^{n}k^m=\frac{B_{m+1}(n)-B_{m+1}}{m+1},$$

where the Bernoulli number B_ℓ is defined by the following expansion

$$\frac{xe^x}{e^x-1}=\sum_{\ell=0}^{\infty}B_\ell\frac{x^\ell}{\ell!}$$

and the Bernoulli polynomial $B_\ell(t)$ is defined by

$$B_\ell(t)=\sum_{j=0}^{\ell}\binom{\ell}{j}B_jt^{\ell-j}.$$

Now, if we let

$$(x+1)(x+2)\cdots(x+m-1)=\sum_{j=0}^{m}\sigma(m,j)x^j, \tag{7}$$

then we may rewrite the left hand side of (3) as

$$\sum_{j=0}^{m}\frac{\sigma(m,j)}{m!}C(j)$$

with

$$C(j)=\sum_{k=1}^{N}k^j.$$

We may therefore retrieve formulas for consecutive powers from Theorem 1.1 by rewriting (3) as

$$\sum_{\ell=0}^{m}\frac{\sigma(m,\ell)}{m!}C(m)=\frac{(N)_{m+1}}{(m+1)!}.$$

For example, for $m=2$, we find that

$$\frac{\sigma(2,2)}{2!}C(2)+\frac{\sigma(2,1)}{2!}C(1)+0=\frac{n(n+1)(n+2)}{6}$$

and we obtain (2) after simplifications.

Remark 1.3. The numbers $\sigma(n,k)$ are known as the Stirling numbers of the first kind or the Stirling cycle numbers [3, p.93]. Note that by definition, for all positive integers m,

$$\sigma(m,0)=0.$$

By multiplying $(x+m)$ to both sides of (7) and using (7) with m replaced by $m+1$, we conclude that for $m\geq 1$ and $j\geq 1$,

$$\sigma(m+1,j)=m\sigma(m,j)+\sigma(m,j-1).$$

2. The q-analogue of (3)

By rewriting

$$\frac{(k)_m}{m!}=\binom{k+m-1}{m},$$

identity (3) can be written as

$$\sum_{k=1}^{n}\binom{k+m-1}{m}=\binom{n+m}{m+1}. \tag{8}$$

In this form, the identity is known and there are three proofs given as exercises in [4, p. 41, Problem 8].

Let $(x;q)_0=1$ and for integer $k\geq 1$, define

$$(x;q)_k=\prod_{j=0}^{k-1}(1-xq^j) \quad \text{and} \quad \begin{bmatrix} n \\ m \end{bmatrix}=\begin{cases}\dfrac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} & \text{if } 0\leq m\leq n,\\ 0 & \text{otherwise.}\end{cases}$$

A q-analogue of (8) is [1, Theorem 3.4, (3.3.9)]

$$\begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}=\sum_{j=0}^{m}q^j\begin{bmatrix} m+j \\ m \end{bmatrix} \quad \text{for } m,n\geq 0,$$

or equivalently,

$$\sum_{j=0}^{n}q^j\frac{(q;q)_{m+j}}{(q;q)_m(q;q)_j}=\frac{(q;q)_{n+m+1}}{(q;q)_{m+1}(q;q)_n}. \tag{9}$$

Identity (9) is a q-analog of (8) because when $q\to 1^-$, (9) reduces to (8).

Similarly, replacing j by $j-1$ in the summation index on the left and setting $N=n+1$ in (9), we find that

$$\sum_{j=1}^{N}q^{j-1}\frac{(q;q)_{m+j-1}}{(q;q)_m(q;q)_{j-1}}=\frac{(q;q)_{N+m}}{(q;q)_{m+1}(q;q)_{N-1}},$$

which is equivalent to

$$\sum_{j=1}^{N}q^{j-1}\frac{(q^j;q)_m}{(q;q)_m}=\frac{(q^N;q)_{m+1}}{(q;q)_{m+1}}. \tag{10}$$

Identity (10) is a q-analog of (3).

We give a proof of (10).

Proof. Multiplying both sides of (10) by $(q;q)_{m+1}$, we see that it suffices to prove

$$\sum_{j=1}^{N}(1-q^{m+1})q^{j-1}(q^j;q)_m = (q^N;q)_{m+1}. \tag{11}$$

Expanding the left side, we find that

$$\begin{aligned}\sum_{j=1}^{N}\left\{q^{j-1}(q^j;q)_m - q^{m+j}(q^j;q)_m\right\} &= \sum_{j=1}^{N}\left\{-(1-q^{j-1})(q^j;q)_m + (1-q^{m+j})(q^j;q)_m\right\}\\ &= \sum_{j=1}^{N}\left\{-(q^{j-1};q)_{m+1} + (q^j;q)_{m+1}\right\},\end{aligned}$$

which is a telescoping sum that simplifies to $(q^N;q)_{m+1}$.

□

We note that (11) can also be derived using the method of the fourth proof of previous section.

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