How Many "Friday the 13th"s Can There Be in A Calendar Year?

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Being mathematically prepared can enable us to contribute to and benefit from our increasingly digitalized and innovation-driven economy. Rich mathematical tasks afford us the opportunities to develop important mathematical thinking dispositions. We will discuss an example of a rich mathematical task that was motivated by a recent news article on the Paris attacks that occur on 13th November 2015, a Friday.

The task (adapted from https://nrich.maths.org/610) is presented as follows:

> Friday the 13th is considered an unlucky day in Western Superstition. It occurs when the 13th day of the month in the Gregorian calendar falls on a Friday, which **happens at least once every year** but **can occur up to three times in the same year** (https://en.wikipedia.org/wiki/Friday_the_13th).
>
> Without referring to the calendar, how can we explain this fact mathematically?

To start off, we need to know some useful facts:

1. For the common year, there are 28 days in the month of February; 30 days in each of the months of April, June, September, and November; 31 days in each of the rest of the months. For the leap year, the only difference is there are 29 days in the month of February.
2. The 13th day of the month falls on a Friday if its first day falls on a Sunday.
3. The first day of the calendar year (common or leap) can fall on any one of the days in the week from Monday to Sunday.

Let x_i be the cumulative sum of the number of days up to month i, $i = 1,2, \ldots 11$

For the common year, we have $x_1 = 31, x_2 = 59, x_3 = 90, x_4 = 120, x_5 = 151, x_6 = 181, x_7 = 212, x_8 = 243, x_9 = 273, x_{10} = 304, x_{11} = 334$

For the leap year, we have $x_1 = 31, x_2 = 60, x_3 = 91, x_4 = 121, x_5 = 152, x_6 = 182, x_7 = 213, x_8 = 244, x_9 = 274, x_{10} = 305, x_{11} = 335$

For any year where 1st January falls on Monday, we will need the last day of month i to fall on a Saturday in order for the first day of month $i + 1$ to fall on a Sunday.

i.e. $x_i \equiv 6\ (mod\ 7) \Leftrightarrow x_i + 1 \equiv 0\ (mod\ 7)$

$a \equiv b\ (mod\ 7)$ means the difference $a - b$ is integrally divisible by 7.

For any year where 1st January falls on Tuesday, without loss of generality, consider the case where the last day of the first month would fall on a Thursday instead of a Wednesday. To account for the displacement of the first count from Monday to Tuesday, we need to consider $x_1 + 1 \equiv 4\ (mod\ 7)$ instead of $x_1 \equiv 3\ (mod\ 7)$. Hence, for the first day of month $i + 1$ to fall on a Sunday, we need to check if $x_i + 2 \equiv 0\ (mod\ 7)$.

Table 1 shows the congruent modular 7 conditions to check for determining the month in which Friday the 13th falls in any year. We can implement this scheme using Microsoft Excel to determine the month(s) in which Friday the 13th falls for any year. See Figure 1 for an extract of the results in the common year.

Table 1: Congruent modular 7 conditions to check for the respective day that 1st January falls on

Day in which 1st January falls on	Congruent Modular 7 conditions to check
Monday	$x_i + 1 \equiv 0 \ (mod\ 7)$
Tuesday	$x_i + 2 \equiv 0 \ (mod\ 7)$
Wednesday	$x_i + 3 \equiv 0 \ (mod\ 7)$
Thursday	$x_i + 4 \equiv 0 \ (mod\ 7)$
Friday	$x_i + 5 \equiv 0 \ (mod\ 7)$
Saturday	$x_i + 6 \equiv 0 \ (mod\ 7)$
Sunday	$x_i \equiv 0 \ (mod\ 7)$

	A	B	C	D	E	F	G	H	I	J	K	L
1		1st Jan (Mon)			1st Jan (Tue)			1st Jan (Wed)			1st Jan (Thur)	
2	x_i	x_i + 1 ≡ 0 (mod 7)			x_i + 2 ≡ 0 (mod 7)			x_i + 3 ≡ 0 (mod 7)			x_i + 4 ≡ 0 (mod 7)	
3	31	4			5			6			0	1st Feb Sun, 13th Feb Fri
4	59	4			5			6			0	1st March Sun, 13th March Fri
5	90	0	1st April Sun, 13th April Fri		1			2			3	
6	120	2			3			4			5	
7	151	5			6			0	1st June Sun, 13th June Fri		1	
8	181	0	1st July Sun, 13th July Fri		1			2			3	
9	212	3			4			5			6	
10	243	6			0	1st Sept Sun, 13th Sept Fri		1			2	
11	273	1			2			3			4	
12	304	4			5			6			0	1st Nov Sun, 13th Nov Fri
13	334	6			0	1st Dec Sun, 13th Dec Fri		1			2	
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Figure 1: An extract of the Implementation scheme in Microsoft Excel

Table 2 provides a summary for the number of Friday the 13th for any year. We have shown for the common year or the leap year, Friday the 13th happens at least once every year and can occur up to three times in the same year.

Table 2: Frequency of Friday the 13th for the common and leap years

Day in which 1st January falls on	Frequency of Friday the 13th for the Common Year	Frequency of Friday the 13th for the Leap Year
Monday	2 (13th April, 13th July)	2 (13th Sept, 13th Dec)
Tuesday	2 (13th Sept, 13th Dec)	1 (13th June)
Wednesday	1 (13th June)	2 (13th Mar, 13th Nov)
Thursday	3 (13th Feb, 13th Mar, 13th Nov)	2 (13th Feb, 13th Aug)
Friday	1 (13th Aug)	1 (13th May)
Saturday	1 (13th May)	1 (13th Oct)
Sunday	2 (13th Jan, 13th Oct)	3 (13th Jan, 13th April, 13th July)

For another example of a rich mathematical task that engages our students in the Sieve of Eratosthenes, see https://drive.google.com/file/d/1R5uHW-f4cxdaArlehMvapsyFMYXg6JQ9/view