The Least Upper Bound

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For a positive integer n, let $x_1 < x_2 < \cdots < x_n$ be real numbers. x_n is the maximum: $x_k < x_n$ for $k = 1, \ldots, n-1$, or equivalently, $x_k \leq x_n$ for every k.

It is different with infinite sequences. Let $x_k = \frac{k}{k+1}$ for any positive integer k. Since $x_1 < x_2 < \cdots$, there is no maximum. However, $x_k \leq 1$ for every k. We say 1 is an upper bound of x. Clearly, any number larger than 1 is also an upper bound. But a number $r < 1$ cannot be an upper bound, since $x_k > r$ if k is large enough. Thus, the upper bounds of $x = \{x_1, x_2, \ldots\}$ are the real numbers greater than or equal to 1, for which the minimum is 1. We say the least upper bound of x is 1. We look upwards through a sequence for the maximum, but from above a sequence look downwards for the least upper bound.

We digress to a task about the sequence x.

Let $r < 1$ be positive. Find a positive integer K such that $x_k > r$ for every $k \geq K$.

Solution. We work backwards. If $x_k > r$, then $1 - r > \frac{1}{k+1}$, and $k + 1 > \frac{1}{1-r}$. So let K be an integer with $K + 1 > \frac{1}{1-r}$. Then $x_K > r$, and for every $k > K, x_k > x_K > r$.

Now we present a general definition of an upper bound.

Let S be a nonempty set of real numbers: $S \subseteq \mathbb{R}$. A real number u is an upper bound of S if $s \leq u$ for every $s \in S$.

An upper bound of a sequence is really an upper bound of the set of values in the sequence. For a strictly increasing sequence such as x above, the distinction is immaterial. For the alternating sequence $\{1, -1, 1, -1, \ldots\}$, the set of values is $\{1, -1\}$: the difference is unimportant as far as upper bounds are concerned.

Here is an important fact about $\mathbb{R}$.

Least Upper Bound Property *If a nonempty $S \subseteq \mathbb{R}$ has an upper bound, then the set of upper bounds has a minimum, called the least upper bound or supremum of S.*

In order to appreciate the seemingly obvious conclusion, note that an infinite set may not have a minimum. For instance, the open interval from 0 to 1. But if S is bounded above, it is guaranteed that the infinitely many upper bounds have a minimum. If S

has a maximum, then the supremum is the maximum. For example, the maximum and supremum of $\{\frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, 1, \ldots\}$ are 1.

Analogously, ℓ is a lower bound of S if $s \geq \ell$ for every $s \in S$. If S is bounded below, then it has a greatest lower bound or infimum. If S has a minimum, then the infimum is the minimum.

We go back to $x = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$, which gets closer and closer (converges) to 1: we say the limit of x is 1. In contrast, $\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots\}$ does not have a limit, since it hops between and closer to -1 and 1. The definition of a limit is a mathematical milestone, and its spirit is encapsulated in the task about x. Using the definition, the following fact can be deduced.

If an increasing sequence $x_1 \leq x_2 \leq \cdots$ is bounded above, then it has a limit, which is its supremum.

Therefore, an increasing sequence of real numbers does one of two things. Either it is bounded above, in which case it converges to its supremum, or it is not bounded above, in which case we say it converges to infinity. Similarly, a decreasing sequence converges to its infimum or negative infinity.