Graph Theory Problems/Solns

1. There are n participants in a meeting. Among any group of 4 participants, there is one who knows the other three members of the group. Prove that there is one participant who knows all other participants.

Soln. Define a graph where each vertex corresponds to a participant and where two vertices are adjacent iff the two participants they represent know each other. Take a vertex a of maximum degree. We claim that this vertex is adjacent to every other vertex. Suppose, on the contrary, that there is a vertex b which is not adjacent to a. Then every pair of vertices in the neighbourhood N(a) of a are adjacent. Furthermore, at least one of the vertices in N(a) is adjacent to b. The degree of this vertex is larger than that of a, a contradiction.

(Note: It's not hard to see that in fact at least n-3 vertices have this property (see Problem 12.

2. In a group of 2n people, $n \ge 2$, each one knows at least n other people. Prove that in this group, there are four people who can be seated at a round table so that so that each person knows both his neighbours.

Soln. Define a graph where each vertex corresponds to a person and where two vertices are adjacent iff the two people they represent know each other. We need to prove that there is 4 cycle in the graph. If the graph is complete, then there is a 4-cycle. If the graph is not complete, then there is pair of vertices, say a, b, which are nonadjacent. Since |N(a)| and |N(b)| are both $\geq n$ and there are only 2n - 2 vertices other than a and b, $|N(a) \cap N(b)| \geq 2$. Let $c, d \in N(a) \cap N(b)$. Then the four vertices a, b, c, d form a 4-cycle.

3. There are n people in a gathering. Some of them are mutual friends. Prove that it is possible to divide them into two groups so that each person has at least half of his friends in a different group.

Soln. Choose A and B such that $A \cup B = V$ and that the number of edges joining a vertex in A to a vertex in B is maximized.

4. In a group of 100 people, each one knows at least 67 other people. Prove that there exist 4 people who are mutual friends.

Soln. For any vertex a, $|N(a)| \ge 67$. Thus the number of vertices not in N(a) is at most 32. Let $b \in N(a)$. Then $|N(b) \cap N(a)| \ge 34$. Let $c \in N(a) \cap N(b)$. Since there are only 66 vertices which are not in $N(a) \cap N(b)$, c is adjacent to at least one vertex, say d, in $N(a) \cap N(b)$. The four vertices a, b, c, d form a K_4 .

5. Suppose you want to modify the above problem by changing the number 100 to 1000. How should 67 be altered so that the conclusion remains the same?

Soln. Ans: 667.

6. There are 500 participants in a conference. Every participant has 400 friends. Is it possible to find a group of 6 mutual friends?

Soln. Construct a graph where each vertex represents a participant and two vertices are adjacent iff the corresponding participants are mutual strangers. The graph consisting of 5 disjoint copies of K_{100} shows that the answer is no.

7. What is your answer if everyone has 401 friends?

Soln. The solution is similar to the previous problem.

8. Prove that in a group of 18 people, there is either a group of 4 mutual friend or a group of 4 mutual strangers.

Soln. Consider K_{18} and colour its edges using two colours, red and blue. At any vertex a, there are 9 edges of the same colour, say ab_1, \ldots, ab_9 are red. Since in K_9 there is either a red C_3 , which together with a will give a red K_4 or a blue K_4 .

9. There are 18 contestants in a tournament. In each round the contests are paired and play each other once. Prove that after 8 rounds, there are three contestant who have not played against each other.

Soln. Consider the graph formed when two vertices are joined by a edge iff they have not played each other. Thus each vertex is of degree 9. We need to prove that in this graph there is a 3-cycle. Suppose on the contrary that there are no 3 cycles. Let the vertices be $a_1, \ldots, a_9, b_1, \ldots, b_9$. Let the neibhgours of a_1 be b_1, \ldots, b_9 . Since there are no C_3 , the neighbours of b_1 must be a_1, \ldots, a_9 . The same goes for b_2, \ldots, b_9 . Thus in this graph each a_i is adjacent to b_1, \ldots, b_9 and each b_i is adjacent to a_1, \ldots, a_9 . (This is known as a complete bipartite graph.) Thus in each round, the players labeled a_1, \ldots, a_9 are paired. But this is impossible. So there must be a 3-cycle.

10. Let G be a graph with 10 vertices. Among any three vertices of G, at least two are adjacent. Find the least number of edges that G can have. Find a graph with this property.

Soln. The answer is 20. An example is the graph consisting of two copies of K_5 . To prove that the answer is 20, let G be a graph with the prescribed property. If a is a vertex with deg $a \leq 2$, then the vertices outside N(a) must induced a subgraph which is complete and hence has more than 20 edges. If a a vertex with deg a = 3, then the vertices outside a induces a complete graph on 6 vertices and has 15 edges. Let $b, c, d \in N(a)$. There is an edge in each of $\{b, c, x\}$, $\{b, d, y\}$ and $\{c, d, z\}$, where x, y, z are three distinct vertices not in $N(a) \cup a$. Thus there are at least 21 edges. Thus every vertex of G is of degree at least 4 and hence G has at least 20 edges.

11. In an $n \times n$ matrix, the rows are pairwise distinct. Prove that there is a column, whose removal results in an $n \times (n-1)$ matrix with the same property.

Soln. Let v_1, \ldots, v_n represent the *n* rows of the matrix. Suppose the conclusion does not hold, then after deleting the first column, two rows will be identical. Join the two rows by an edge and label it 1. After that replace column 1. Do the same for all the other columns. We will have *n* edges labeled $1, 2, \ldots, n$. The graph contains at least one cycle, say $a_1, a_2, \ldots, a_k, a_1$ with the edge $a_i a_{i+1}$ labeled as t_i . Then a_1 and a_2 differ only in column t_1, a_2 and a_3 differ only in column t_2 . Thus a_1 and a_3 differ in columns t_1, t_2 . Continuing this way, we know that a_1 and a_k differ in columns t_1, \ldots, t_{k-1} . We also know that these two rows differ only in column t_k . This leads to a contradiction.

12. In a group of 1997 people, among 4 of them there is at least one who knows the other three. What is minimum of people in the group who knows everybody else?

Soln. Construct a a graph in which each person is a vertex and two vertices are adjacent if the corresponding persons know each other. Then the question asks for the minimum number of vertices with degree 1996. If every pair of vertices are adjacent, then every vertex is of degree 1996.

If a and b are not adjacent, then c and d are adjacent. If a and b are both adjacent to every vertex in the graph, then there are 1995 vertices with degree 1996. If a and c are not adjacent, then each of a, b, c is adjacent to every other vertex in the graph. Thus there are 1994 vertices of degree 1996. Thus the answer is 1994.

13. At the end of a birthday party, the hostess wants to give away candies. She has 6 types of cookies. Each child is given a gift packet which contains two types of cookies. Each type of cookie is used is combination with at least three others. Prove there are three children, who between them, have all the six types of cookies.

Soln. Form a graph with each type of candies corresponding to a vertex. Two vertices are joined by an edge if the corresponding types of candies are used together in a gift pack. In this graph every vertex is of degree ≥ 3 . To solve the problem, we need to show that the graph contains three edges which are pairwise nonadjacent (such a set of edges are said to be independent.). Let a be a vertex and b, c, d be 3 of its neighbours. Let the remaining two vertices be e, f (these may also be neighbours of a). Finally, let $A = \{a, b, c, d\}$ and $B = \{b, c, d\}$. Note that $|N(e) \cap A| \geq 2$ and $|N(f) \cap A| \geq 2$. If $|(N(e) \cup N(f)) \cap B| \geq 2$, then there exists 2 vertices in B, say b and c, such that be and cf are edges. Then be, cf and ad are 3 independent edges. If $|(N(e) \cup N(f)) \cap B| = 1$, say b is the common neighbour of e and f, then e and f are both adjacent to a. Since the degree of d is ≥ 3 , and d is not adjacent to e, f, d must be adjacent to c and b. Thus ae, bfcd are 3 independent edges.

14. In a grooup of people, any two mutual friends have no common friends while any pair of mutual strangers have exactly two common friends. Prove that there are two persons in this group who have the same number of friends.

Soln. Construct a graph where each vertex represents a person and two vertices are adjacent if the two corresponding persons are friends. For any vertex a, the vertices in N(a) are pairwise nonadjacent. Let b be a vertex adjacent to a. Then the two sets $N(a) - \{b\}$, $N(b) - \{a\}$ are disjoint. Moreover, for any vertex $x \in N(a) - \{b\}$, there is

unique vertex $y \in N(b) - \{a\}$ such that y and a are the common neighbours of b and x. This sets up a 1-1 correspondence and proves that |N(a)| = |N(b)|.

15. In a party there are 12k guests. Every guest knows exactly 3k + 6 other guests. Suppose that if x knows y, then y knows x too. For every two guests x and y in this party there are exactly n guests who know both x and y. (n is a constant). Prove that

$$9k^2 + (33 - 12n)k + (30 + n) = 0$$

and then solve for n and k.

Soln. There are v = 12k vertices and $\deg(x) = 3k + 6$ for every vertex x. For every pair of vertices x, y, define $\deg(x, y)$ to be the number of vertices adjacent to both x and y. Thus $\deg(x, y) = n$. Since there are $\binom{12k}{2}$ pairs of vertices, we have $\sum \deg(x, y) = n\binom{12k}{2}$. Since $\deg(x) = 3k + 6$ for every vertex, we have $\sum \deg(x, y) = 12k\binom{3k+6}{2}$ and hence $12k\binom{3k+6}{2} = n\binom{12k}{2}$. This gives the desired equation. We can solve the equation as follows: From the equation, one concludes that n is divisible by 3 and by 2. Thus it is a multiple of 6. Consider the equation as a quadratic equation in k. Since its solution is an integer, we have

$$(33 - 12n)^2 - 36(30 + n) = m^2$$

for some integer m. Since m is a multimple of 3, we can write m = 3m'. Dividing throughout by 9, we have

$$16n^2 - 92n + 1 - {m'}^2 = 0.$$

By completing square, we have

$$(8n-23)^2 - (2m')^2 - 525 = 0$$
, or $(8n-23-2m')(8n-23+2m') = 525$.

Thus in any factorization of 525 into two factors $a \times b$, we must have a + b = 16n - 46. Since *n* is multiple of 6, we see that a + b must leave a remainder of 2 when divided by 3. There are 6 factorization of $525 = 3 \times 7 \times 5^2$:

$$1 \times 525, 3 \times 175, 5 \times 105, 7 \times 75, 15 \times 35, 21 \times 25.$$

Only 5×105 and 15×35 have this property. The first gives n = 39 and the second gives n = 6. Consequently, n = 6 and k = 3.

16. Given n points on the plane such that the distance between every pair of points is at least 1, prove that there are at most 3n pairs of points such that the distance between two points in each pair is 1.

Soln. Take the points as vertices and a pair of vertices are joined by an edge iff their distance apart is 1. At any point draw a circle of radius 1. Since the points are of distance at least 1, there are at most six points on the circle. Thus the degree of every vertex is at most 6 and there are at most 6n/2 = 3n edges.

17. There are 9 mathematicians in a meeting. It was discovered that each of them can speak at most three languages and among any three of them at least two can speak a common language. Prove that three are three mathematicians who speak a common language.

Soln. Construct a graph with each mathematician as a vertex and two vertices are joined by an edge iff the corresponding mathematicians speak a common language. If there is vertex a with deg a = 4 since a speaks at most three languages, two of the neighbours of a must speak a language in common with a and we are done. Otherwise the degree of every vertex is at most 3. Consider a vertex a, there are at least 5 vertices which are not adjacent to a, and suppose that b is one of them. Since b has at most three neighbours, there is vertex, say c, outside $N(a) \cup N(b)$. Thus a, b, c are three vertices with no edge joining any two of them, a contradiction. Thus the second case is impossible.

18. Can you place one number chosen from $\{0, 1, \ldots, 9\}$ on each of the vertices of polygon with 45 sides, so that for every pair of of integers, $a, b, 0 \le a < b \le 9$, there is a side of the polygon whose ends have the numbers a and b?

Soln. Each number must appear 5 times. So it is impossible.

19. What is your answer if in the previous problem, the numbers 45 and 9 are replace by 55 and 10?

Soln. Yes. Place the numbers in the order

 $0, 1, 2, \dots, 10, 0, 2, 4, \dots, 9, 0, 3, 6, \dots, 8, 0, 4, 8, \dots, 7, 0, 5, 10, \dots, 6.$

20. There are 3 schools each with n students. Every students knows n + 1 students from the other two schools. Prove that it is possible to find one student from each of the schools such that the three students know each other.

Soln. Let the schools be A, B, C. Represent each student by a vertex and a pair of vertices are joined by an edge iff two corresponding students know each other. For each vertex $x \notin A$, define $N_A(x)$ to be the set of vertices in A which are adjacent to x. Let $m = \max\{|N_Y(x)| : x \notin Y\}$ and let $|N_A(b)| = m$ where $b \in B$. Then b is adjacent to m vertices in A and at least one vertex, say c, in C. Since $|N_B(c)| \leq m$, we have $|N_A(c) \geq n + 1 - m$. But $|N_A(b)| = m$. Thus $N_A(c) \cap N_A(b)$ must contain at least one vertex, say a. Then a, b, c know each other.

21. There are *n* people in a room. Any group of $m \ge 3$ people in the room have a unique common friend. Can you determine *n* in terms of *m*?

Soln. Construct a graph in the naturally way. If the graph contains K_i as a subgraph for some $i \leq m$, then every vertex of this subgraph is adjacent to a common vertex. Thus it also contains a K_{i+1} . From the hypothesis, the graph contains a K_2 . Thus it contains a

 K_{m+1} . Let $A = \{a_1, \ldots, a_{m+1}\}$ be the vertex set of this K_{m+1} . Suppose there is a vertex $b \neq a_i, i = 1, \ldots, m+1$ and that b is adjacent to two vertices, say a_m, a_{m+1} , in A. Then a_m and a_{m+1} are the common neighbours of the m vertices b, a_1, \ldots, a_{m-1} , a contradiction. So there are m vertices, say a_1, \ldots, a_m , in A which are not adjacent to b. The m vertices b, a_3, \ldots, a_{m+1} have a common neighbour, say c. Now $c \neq a_i, i = 1, \ldots, m+1$ but c is is adjacent to $m - 1 \geq 2$ vertices in A, which is impossible. Thus the graph is just K_{m+1} and n = m + 1.

22. (IMO 1990) Let $n \ge 3$ and consider a set E of 2n - 1 distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

Soln. Label the points $0, \ldots, 2n-2$. E is good if it contain 2 points a and b such that $|a-b| \equiv n-2 \pmod{2n-1}$. Consider the graph G where $\{a,b\}$ is an edge iff $|a-b| \equiv n-2 \pmod{2n-1}$. Since each vertex if of degree 2, G is a union of d cycles each of length $\lfloor \frac{2n-1}{d} \rfloor$, where $d = \gcd(n-2, 2n-1)$. Note that d = 3 if 3 divides 2n-1 and d = 1 otherwise. A maximal E which is not good contains $\lfloor \frac{2n-1}{2d} \rfloor$ from each cycle. Thus a minimal good E contains

$$d\left\lfloor\frac{2n-1}{2d}\right\rfloor + 1 = \begin{cases} n-1 & \text{if } n \equiv 2 \pmod{3} \\ n & \text{otherwise} \end{cases}$$

points.

23. Find the smallest positive integer n such that in any set of n irrational numbers, there are three numbers such that the sum of every two of them is again irrational.

Soln. If n = 4, then in any three of the four numbers $\pm\sqrt{2}$, $1 \pm \sqrt{2}$, there are two whose sum is rational. Thus $n \ge 5$. Now consider any set of 5 irrational number. Construct a graph G whose vertices corresponds to the numbers such that two vertices are adjacent iff the corresponding numbers sum to a rational number. Then G contains no C_3 for if x + y, y + z, z + x are rational, then x, y, z are all rational. Similarly, G does not C_5 . These imply that the graph is bipartite. One of the partite sets contains three vertices a, b, c, say. Then then sum of any 2 of them remains irrational.

(Note: If you don't know bipartite graphs then you can proceed as follows: If there is a vertex of degree at least three, say a is adjacent to b, c, d, then b, c, d are pairwise nonadjacent. If there is a vertex of degree 1 say x, then among the three vertices which not adjacent to a, there two, say y, z which are nonadjacent. Then x, y, z are three pairwise nonadjacent vertices. If every vertex if of degree two, then the graph is a C_5 , which is impossible.)

24. (IMO shortlist 2002) Let n be an even integer. Show that there exists a permutation x_1, x_2, \ldots, x_n of $1, 2, \ldots, n$ such that x_{i+1} is one of $2x_i, 2x_i - 1, 2x_i - n, 2x_i - n - 1$ (take $x_{n+1} = x_1$).

Soln. Let n = 2m. When $x_i \le m$, $2x_i - n < 0$ and $2x_i - n - 1 < 0$. Thus x_{i+1} is either $2x_i$ or $2x_i - 1$. When $x_i > m$, $2x_i > n$ and $2x_i - 1 > n$ and thus x_{i+1} is either $2x_i - n$ or $2x_i - n - 1$. This means that for any x_i , x_{i+1} is one of the two numbers 2k - 1 and 2k for some $k \in \{1, \ldots, m\}$.

Conversely, for any $x_i = 2k$ or 2k - 1, x_{i-1} one of two numbers k and k + m, where $k \in \{1, \ldots, m\}$, taking $x_0 = x_n$.

We now construct a graph with m vertices as follows. The vertex v_i represents the two numbers 2i - 1 and 2i. We draw two arrows from v_i : e_{2i} to the vertex that represents the possible value(s) of x_{i+1} when $x_i = 2i$, and e_{2i-1} to the vertex that represents the possible value(s) of x_{i+1} when $x_i = 2i - 1$. These arrows may join a vertex to itself; it does not matter.

Then there will be exactly two arrowheads and two arrowtails connected to each vertex. Choose an arbitrary vertex. Follow the arrows in order, i.e. leaving a vertex by an arrowtail and reaching a vertex by an arrowhead. Because each vertex is connected to exactly two arrowheads and two arrowtails, it must be possible to form an eulerian trail, say $e_{y_1}, e_{y_2}, \ldots, e_{y_n}$, following the arrows in order if the graph is connected. Then y_1, y_2, \ldots, y_n is the required permutation.

We now prove that the graph is connected. v_1 and v_2 are connected because $4 = 2 \times 2$. We assume that $v_1, v_2, \ldots, v_{i-1}$ are connected. But now since $2i = 2 \times i$, v_i is connected to the vertex representing *i* which is v_j for some j < i. Thus by the induction hypothesis the graph is connected.

Graph colourings Problems/Solns

1. Given 6 lines in space such that no three lie on the same plane, prove that there exist three lines which satisfy one of the following:

- (i) they are pairwise skew;
- (ii) they are parallel;
- (iii) they are concurrent.

Soln. Represent the six lines by the 6 vertices in K_6 . Colour an edge red and blue if its two incident vertices represent a pair of skew and coplanar lines, respectively. There is always a monochromatic triangle. If the triangle is red, then there are three pairwise skew lines. If the triangle is blue, then either case (ii) or case (iii) will happen. This can be shown as follows. Two of lines, say A and B, are either parallel of concurrent. If A and B are parallel, the third line C, being coplanar with both A and B, can only be parallel to them and we have case (ii). If A and B are concurrent, C being coplanar with both Aand B can only be concurrent with each of them. Thus we have case (iii).

2. Given 6 points in the plane, no three collinear, prove that there are two triangles whose vertices are among the 6 given points such that the longest of one of them is the shortest side of the other.

Soln. Form a graph in the normal way and label the vertices a, b, c, d, e, f. Colour the longest side of every triangle red. (If there are edges of equal length, then we order them once and for all in an arbitrary manner.) If there is a red triangle, then the shortest side of this triangle is the longest side of another triangle. Thus it suffices to prove that there is a red triangle. If there is vertex a which is incident to three red edges, say ab, ac, ad are all red. Then one of the edges, say bc, in the triangle bcd must be red yielding a red triangle abc. Thus it also suffices to prove that there is a vertex which is incident to three red edges.

Now consider the shortest side, say ab. The two vertices a, b together with each of the remaining 4 vertices, form 4 triangles. Thus there are 4 red edges incident to a or b. If at least three of these four red edges are incident to a, then we know that a red triangle will be form. We now suppose that ac, ad, be, bf are red. Consider bcd. If the longest side of this triangle is cd, then acd is a red triangle. If either bc or bd is the longest side, then there are three red edges incident to b. Again a red triangle will be form. Thus there is always a red triangle. The shortest of this red triangle is the longest side of another triangle because every red edge is the longest side of some triangle.

3. Prove that any colouring of the edges of K_6 using two colours produces two monochromatic triangles.

Soln. Call a pair of adjacent edges a monochromatic pair if they have the same colour. There are at least 4 monochromatic pairs at each vertex, giving a minimum of 24 such pairs. On the other hand each monochromatic triangle has three such pairs while a triangle which is not monochromatic has only one such pair.

4. Prove that among a group of 9 people there are 3 mutual friends or there are 4 mutual strangers. (This means $r(3,4) \leq 9$.)

Soln. We need to prove that if we colour the edges of K_9 either red or blue, then there is either blue K_3 or a red K_4 . Suppose there is vertex say a which is incident to at least 4 blue edges, say ab_1, \ldots, ab_4 . If there is a blue edge joining a pair of vertices among b_1, \ldots, b_4 , then there is blue K_3 . Otherwise b_1, \ldots, b_4 form a red K_4 .

Now we consider the case where every vertex is incident to at most three blue edges. We can't have every vertex incident to exactly three blue edges (because the number of vertices of odd degree in any graph must be even.) Thus there is a vertex, say c, which is incident to 6 red edges, say cd_1, \ldots, cd_6 . Then among d_1, \ldots, d_6 , there is either a blue K_3 or a red K_3 . Thus there is either a blue K_3 or a red K_4 . **5.** Prove that r(3, 4) = 9.

Soln. In K_8 , colour the edges of a C_8 together with its main diagonals red and the rest blue.

6. Prove that among a group of 14 people there are 3 mutual friends or there are 5 mutual strangers.

Soln. Similar to 4.

7. Colour the integral points (x, y), where $1 \le x \le 16$ and $1 \le y \le 9$, using three colours. Prove that there exists a monochromatic rectangle whose sides are parallel to the axes.

Soln. At least 46 points have the same colour, say red. At least 6 of these points lie on the same horizontal line. Restrict to these 6 vertical columns. If any of the 8 rows has two red points, we are done. Otherwise there are at least 40 points which are either blue or white. By continuing the argument we will eventually get either a white or a blue rectangle.

8. Colour the points on the plane either red or blue. Prove that there exists a monochromatic equilateral triangle whose sides are of length either 1 or $\sqrt{3}$.

Soln. Draw a circle of radius 1. If the points on the circumference have the same colour, we are done. Otherwise two of the points say A and B have different colours. Erect an isosceles triangle ABC with AC = BC = 2. Suppose A and C have different colours, say A is red and C is blue. Let M be the mid point of AC and assume, without loss of generality, that M is red. Let D, E be points such that MDA and MEA are equilateral triangles. Consider the points A, C, D, E and M. If one of D, E is red, we have a red equilateral triangle of side 1. If both D, E are blue, we have a blue equilateral triangle of side 1. If both D, E are blue, we have a blue equilateral triangle of side 1.

9. Colour the points on the plane using three colours. Prove that there are two points with the same colour and are unit distance apart.

Soln. Consider two pairs of equilateral triangles of side 1, ABC, BCD and AEF, EFG. If the vertices of the triangles do not receive distinct colours, we are done. Otherwise D and G are of the same colour. DG can be made to be equal 1.

10. (Home work problem) Let G be a graph with 9 vertices and m edges. Find the smallest m so that in any colouring of the edges of G with one of two colours, there is a monochromatic C_3 . (This is actually Q3 of IMO1991)

Soln. If m = 33, G always contains a K_6 . Let $\{a, b\}, \{c, d\}, \{e, f\}$ be three pairs of nonadjacent vertices in G. (The other pairs are all adjacent.) These 6 vertices need not be distinct. But there are 3, say a, c, e, which are pairwise adjacent. There are also 3 other vertices, say x, y, z. The six vertices a, c, e, x, y, z form a K_6 . Since in any 2 colouring of the edges of K_6 , there is a monochromatic C_3 , G has a monochromatic C_3 .

It is not hard to contruct a graph with m = 32 and a 2 colouring that produces no monochromatic C_3 .

It is not hard to contruct a graph with m = 32 and a 2 colouring that produces no monochromatic C_3 . For example, let G be the graph with $x, a_1, b_1, \ldots, a_4, b_4$ as vertices and with all pairs of vertices adjacent except for $a_i b_i$, $i = 1, \ldots, 4$. By $i \to j$, we mean the set of edges $\{a_i b_j : i = 1, 2, j = 1, 2\}$. Also by $x \to j$, we mean the set of edges $\{xa_j, xb_j : j = 1, 2\}$. Now we colour the following edges red:

$$x \to 1, \quad x \to 2, \quad 2 \to 3, \quad 3 \to 4, \quad 4 \to 1.$$

The other edges are coloured blue. Then there is no monochromatic C_3 . Thus m = 33.