A SHORT NOTE ON A PEDOE'S THEOREM ABOUT TWO TRIANGLES

K. S. Poh

1. About Daniel Pedoe (adapted from [1])

Daniel Pedoe is Professor of Mathematics at the University of Minnesota. Born and educated in England, he received his Ph.D. from Cambridge University in 1937, having spent a year as a Member of the Institute for Advanced Study, Princeton. In 1947 he was awarded a Leverhulme Research Fellowship, and returned to Cambridge to work with Sir William Hodge on 'Methods of Algebraic Geometry' (Cambridge University Press). The three volumes of this highly regarded text have been translated into Russian, and the first two volumes have been reissued by the Cambridge University Press in paperback. Professor Pedoe has held professorships in Sudan and in Singapore, and became resident in the United States in 1962. He is the author of several mathematics books, all of which show his deep interest in geometry. That he has a gift for exposition is shown by the success of 'The Gentle Art of Mathematics , published by Penguin Books, and the award of a Lester R. Ford prize for exposition by the Mathematical Association of America.

2. Motivation

The Pedoe's theorem which we are going to discuss gives an inequality relating the six sides of two triangles and their areas. More precisely, on the greater side we have a symmetric expression relating the six sides of two triangles, and on the samller side we have a symmetric expression relating their areas. Our approach of proving this Pedoe's theorem is as follows: we first make use of Cauchy's Inequality to transform the greater side to an intermediate expression of smaller magnitude, which is also symmetric in terms of the six sides of two triangles but free from interaction between one another, and then prove that this intermediate expression is still greater than or equal to the smaller side in Pedoe's inequality. Moreover, the respective necessary and sufficient condition for each of the equality signs to hold has also been obtained. This intermediate expression has thus become an improved bound of Pedoe's Theorem.

3. Notation and basic lemmas

Throughout this note, ΔABC and $\Delta A'B'C'$ denote two arbitrary triangles. As usual, a, b, and c are the three sides of ΔABC opposite to the three interior angles A, B, and C respectively. Let the area of ΔABC be denoted by Δ. Likewise, a', b', c', A', B', C', and Δ' are defined for $\Delta A'B'C'$.

Next, Σa^2 denotes the sum which is a symmetric expression in which a^2 is a representative term, that is, $\Sigma a^2 = a^2 + b^2 + c^2$. Likewise, we have $\Sigma(a^2 a'^2) = a^2 a'^2 + b^2 b'^2 + c^2 c'^2$, $\Sigma \cot A = \cot A + \cot B + \cot C$, etc.

Throughout this note, the expressions $a'^2(b^2 + c^2 - a^2) + b'^2(c^2 + a^2 - b^2) + c'^2(a^2 + b^2 - c^2)$ and $(\Sigma a^2)(\Sigma a'^2) - 2\sqrt{(\Sigma a^4)(\Sigma a'^4)}$ are denoted by D and E respectively.

Now, we state and prove a special case of Cauchy's inequality. The following proof can be understood by anyone who knows about the scalar product of vectors at G.C.E. 'A' Level.

Lemma 1. $\Sigma(a^2 a'^2) \leqslant \sqrt{(\Sigma a^4)(\Sigma a'^4)}$ and the equality holds if and only if $\Delta ABC \sim \Delta A'B'C'$.

Proof. Consider the two vectors $\underset{\sim}{u} = (a^2, b^2, c^2)$ and $\underset{\sim}{v} = (a'^2, b'^2, c'^2)$ in $\mathcal{R}^3$. We have $\underset{\sim}{u} \bullet \underset{\sim}{v} \leqslant |\underset{\sim}{u}|\,|\underset{\sim}{v}|$ and the equality holds if and only if $\underset{\sim}{u} = \lambda \underset{\sim}{v}$ for some positive constant λ, hence the result.

Lemma 2. $D \geqslant E$ and equality holds if and only if $\Delta ABC \sim \Delta A'B'C'$.

Proof.
$$\begin{aligned} D &= a'^2(\Sigma a^2 - 2a^2) + b'^2(\Sigma a^2 - 2b^2) + c'^2(\Sigma a^2 - 2c^2) \\ &= (\Sigma a^2)(\Sigma a'^2) - 2\Sigma(a^2 a'^2) \\ &\geqslant (\Sigma a^2)(\Sigma a'^2) - 2\sqrt{(\Sigma a^4)(\Sigma a'^4)} \quad \text{(by Lemma 1)} \\ &= E. \end{aligned}$$

Clearly, $D = E$ if and only if $\Delta ABC \sim \Delta A'B'C'$ by Lemma 1.

Lemma 3. $\Sigma \tan A = \tan A \tan B \tan C$.

Proof.
$$\begin{aligned} \Sigma \tan A &= \tan A + \tan B + \tan C \\ &= \tan A + \tan B - \tan(A+B) \quad (\text{as } C = 180° - (A+B)) \\ &= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= (\tan A + \tan B)\left(1 - \frac{1}{1 - \tan A \tan B}\right) \\ &= (\tan A + \tan B) \bullet \left(\frac{-\tan A \tan B}{1 - \tan A \tan B}\right) \\ &= \tan A \tan B \bullet \left(-\frac{\tan A + \tan B}{1 - \tan A \tan B}\right) \\ &= \tan A \tan B \tan C. \end{aligned}$$

Lemma 4. $\Sigma(\cot A \cot B) = 1.$

Proof. $\Sigma(\cot A \cot B) = \Sigma\left(\dfrac{1}{\tan A \tan B}\right)$

$$= \frac{\Sigma \tan A}{\tan A \tan B \tan C}$$

$$= 1 \quad \text{by Lemma 3.}$$

Lemma 5. $\Sigma a^2 = 4\Delta\,\Sigma \cot A.$

Proof. $\Sigma a^2 = (a^2 + b^2 - c^2) + (b^2 + c^2 - a^2) + (c^2 + a^2 - b^2)$

$$= 2ab\cos C + 2bc\cos A + 2ac\cos B \quad \text{(by Cosine Rule)}$$

$$= 4 \bullet \frac{1}{2}ab\sin C\cot C + 4 \bullet \frac{1}{2}bc\sin A\cot A + 4 \bullet \frac{1}{2}ac\sin B\cot B$$

$$= 4\Delta(\cot C + \cot A + \cot B) \quad (\text{as } \Delta = \frac{1}{2}ab\sin C, \text{ etc.})$$

$$= 4\Delta\,\Sigma\cot A.$$

Lemma 6. $\Sigma a^4 = 8\Delta^2[(\Sigma \cot A)^2 \quad 1].$

Proof. $\Sigma a^4 = (\Sigma a^2)^2 - 2\Sigma(a^2b^2),$

$$ab = 2 \bullet \frac{1}{2}ab\sin C\operatorname{cosec} C = 2\Delta\operatorname{cosec} C.$$

$$\begin{aligned}
\text{Thus } \Sigma(a^2b^2) &= 4\Delta^2\ \Sigma\operatorname{cosec}^2 A \\
&= 4\Delta^2\ \Sigma(1 + \cot^2 A) \\
&= 4\Delta^2(3 + \Sigma\cot^2 A) \\
&= 4\Delta^2[3 + (\Sigma\cot A)^2 - 2\Sigma(\cot A\cot B)] \\
&= 4\Delta^2[1 + (\Sigma\cot A)^2] \quad \text{by Lemma 4.}
\end{aligned}$$

$$\begin{aligned}
\text{Hence } \Sigma a^4 &= 16\Delta^2(\Sigma\cot A)^2 - 8\Delta^2[1 + (\Sigma\cot A)^2] \quad \text{(by Lemma 5 and above)} \\
&= 8\Delta^2[(\Sigma\cot A)^2 - 1].
\end{aligned}$$

Lemma 7. $\Sigma\cot A \geqslant \sqrt{3}.$

Proof. If $A = B = C = 60°$, the result is clearly true. Otherwise there exists at least one angle greater than $60°$. Without loss of generality, let $B > 60°$. Construct an equilateral triangle A′BC as shown in Figure 1.

Considering $\Delta AA'B$, we have

$$\begin{aligned}
AA'^2 &= a^2 + c^2 - 2ac\cos(B - 60°) \\
&= a^2 + c^2 - 2ac\left(\cos B \bullet \frac{1}{2} + \sin B \bullet \frac{\sqrt{3}}{2}\right)
\end{aligned}$$

$$= a^2 + c^2 - \frac{1}{2}(a^2 + c^2 - b^2) - 2\sqrt{3} \bullet \frac{1}{2} ac \sin B$$

$$= \frac{1}{2}(a^2 + b^2 + c^2) - 2\sqrt{3}\Delta$$

$$= 2 \Delta \Sigma \cot A - 2\sqrt{3}\Delta \quad \text{(by Lemma 5)}$$

$$= 2 \Delta (\Sigma \cot A - \sqrt{3}).$$

Since $AA'^2 > 0$, we must have $\Sigma \cot A > \sqrt{3}$. The proof is complete.

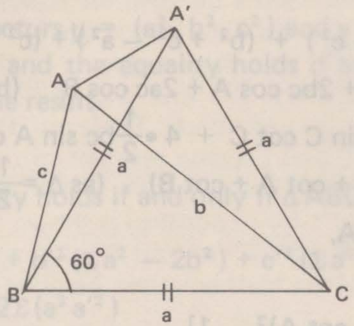

Fig. 1

4. Intermediate expression for Pedoe's Inequality

The original form of Pedoe's Theorem states that for any two triangles ΔABC and $\Delta A'B'C'$, $D \geqslant 16\Delta\Delta'$ and the equality holds if and only if $\Delta ABC \sim \Delta A'B'C'$. We now provide an intermediate expression between D and $16\Delta\Delta'$.

Extended Pedoe's Theorem

Let ΔABC and $\Delta A'B'C'$ be any two triangles. Let Δ and C' denote the areas of ΔABC and $\Delta A'B'C'$ respectively.

Let $D = a'^2 (b^2 + c^2 - a^2) + b'^2 (c^2 + a^2 - b^2) + c'^2 (a^2 + b^2 - c^2)$
and $E = (\Sigma a^2)(\Sigma a'^2) - 2\sqrt{(\Sigma a^4)(\Sigma a'^4)}$.

Then $D \geqslant E \geqslant 16\Delta\Delta'$, and $E = 16\Delta\Delta'$ if and only if $\Sigma \cot A = \Sigma \cot A'$. Moreover, the following are equivalent:

(1) $D = 16\Delta\Delta'$ (2) $D = E$ (3) $\Delta ABC \sim \Delta A'B'C'$.

Proof. By Lemma 2, $D \geqslant E$. Next, by Lemmas 5 and 6, we have

$$E = 16\Delta\Delta'(\Sigma \cot A)(\Sigma \cot A') - 16\Delta\Delta' \sqrt{[(\Sigma \cot A)^2 - 1][(\Sigma \cot A')^2 - 1]}$$

$$= 16\Delta\Delta'[(\Sigma \cot A)(\Sigma \cot A') - \sqrt{[(\Sigma \cot A)^2 - 1][(\Sigma \cot A')^2 - 1]}\,].$$

Since $[(\Sigma \cot A)(\Sigma \cot A') - 1]^2 - [(\Sigma \cot A)^2 - 1][(\Sigma \cot A')^2 - 1]$
$= (\Sigma \cot A - \Sigma \cot A')^2 \geqslant 0,$

we have $[(\Sigma \cot A)(\Sigma \cot A') - 1]^2 \geqslant [(\Sigma \cot A)^2 - 1][(\Sigma \cot A')^2 - 1]$. By Lemma 7, $(\Sigma \cot A)(\Sigma \cot A') - 1 \geqslant \sqrt{3} \bullet \sqrt{3} - 1 = 2 > 0$ and $[(\Sigma \cot A)^2 - 1][(\Sigma \cot A')^2 - 1] \geqslant (3-1)(3-1) = 4 > 0$.

Therefore $(\Sigma \cot A)(\Sigma \cot A') - 1 \geqslant \sqrt{[(\Sigma \cot A)^2 - 1][(\Sigma \cot A')^2 - 1]}$.

Hence $(\Sigma \cot A)(\Sigma \cot A') - \sqrt{[(\Sigma \cot A)^2 - 1][(\Sigma \cot A')^2 - 1]} \geqslant 1$. Thus $E \geqslant 16\Delta\Delta'$, completing the proof of $D \geqslant E \geqslant 16\Delta\Delta'$. Note that $E = 16\Delta\Delta'$ if and only if $\Sigma \cot A = \Sigma \cot A'$. Next, clearly (1) implies (2). By Lemma 2, (2) implies (3). Finally, since (3) implies both (2) and $E = 16\Delta\Delta'$, we have (3) implies (1). The proof is complete.

Remark. As illustrated in the following example, the difference between D and $16\Delta\Delta'$ can be quite large when compared with that between E and $16\Delta\Delta'$. For ΔABC take $a = 3$, $b = 4$, $c = 5$, and for $\Delta A'B'C'$, take $a' = 6$, $b' = 7$, $c' = 8$. Then $D = 2034 > E = 1974 > 16\Delta\Delta' = 1952$.

References

[1] Daniel Pedoe, A Course of Geometry for Colleges and Universities, Cambridge University Press, 1970.

[2] Coxeter, Twelve Geometric Essays, Southern Illinois University Press, 1968.