Letters to the Editor

Heronian Triangles

I refer to the article "Heronian Triangles" which appeared in Mathematical Medley Volume 22 No. 2 September 1995. I would like to give here a construction of infinitely many primitive scalene Heronian triangles which are not right-angled.

Let k be any positive integer. Let $\alpha = 2k^2 + 2k + 1$, $\beta = 2k^2 - 2k + 1$ and $\gamma = (\alpha\beta - 1)(\alpha + \beta)$. Then the triple (a, b, c) where $a = \beta + \gamma$, $b = \gamma + \alpha$, $c = \alpha + \beta$, forms a primitive scalene Heronian triple which is not Pythagorean.

First of all we verify easily that the area of a triangle with sides a, b, c is an integer. It is namely equal to $\alpha\beta(\alpha + \beta)\sqrt{\alpha\beta - 1}$, where $\alpha\beta - 1 = 4k^4$ is a perfect square. Moreover, $a \neq b \neq c \neq a$ since $\alpha \neq \beta \neq \gamma \neq \alpha$, so (a, b, c) is a scalene Heronian triple.

Next, let d be a common divisor of a, b, c. According to the definitions a and b are odd. It follows that d is odd. Now d is also factor of $c + a - b = 2\beta$ as well as $b + c - a = 2\alpha$. Since d is odd, d must then divide both α and β, and thus also $\alpha - \beta$, where $\alpha - \beta = 4k$. Then once again d is a factor of k since d is odd. It follows that d is a common divisor of k and α.. But $gcd(\alpha, k) = 1$ by definition of α. Hence $d = 1$ and (a, b, c) is a primitive triple.

Lastly, we note that $\gamma > \alpha > \beta$, so that b is the longest side of a triangle with sides a, b and c. We have $b^2 - a^2 = (\gamma + \alpha)^2 - (\beta + \gamma)^2 = (\alpha - \beta)(\alpha + \beta + 2\gamma) = 4k(\alpha + \beta + 2\gamma)$; and $c^2 = (\alpha + \beta)^2 = 4(2k^2 + 1)^2$.

Now when $k = 1$, $(a, b, c) = (25, 29, 6)$; and if $k > 1$, k cannot be a factor of $(2k^2 + 1)^2$ (because k and $2k^2 + 1$ are coprime). Thus in any case $b^2 - a^2 = c^2$ cannot hold, so (a, b, c) is not a Pythagorean triple. This completes the proof.

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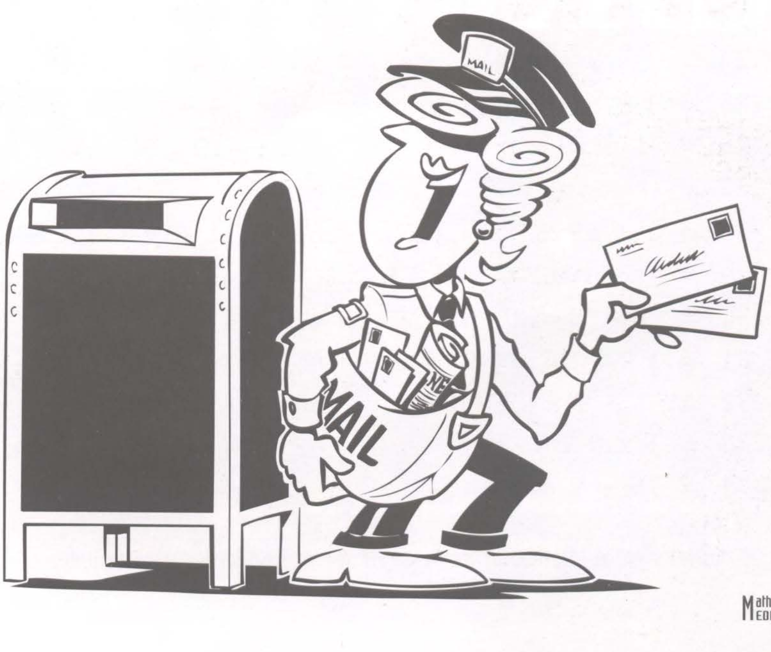

Figure it out...

The item "Figure it out ..." in the last issue of Mathematical Medley (Vol. 22 No. 2 September 1995) was said to be extracted from a project "1995" by two lower secondary students.

Perhaps there are still some young students interested in a little fun with a project "1996". A possible result of such a project is attached.

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$1 = 1 + (9 - 9) \times 6$

$2 = 1 + (\sqrt{9} + \sqrt{9}) \div 6$

$3 = 1 \times (9 + 9) \div 6$

$4 = 19 - 9 - 6$

$5 = -1 - 9 + 9 + 6$

$6 = 1 \times 9 - 9 + 6$

$7 = 1 + 9 - 9 + 6$

$8 = 1 + 9 \div 9 + 6$

$9 = \sqrt{(1 \times 9)} + (9 - 6)!$

$10 = 19 - \sqrt{9} - 6$

$11 = -1 + (9 + 9 - 6)$

$12 = 1 \times 9 + 9 - 6$

$13 = 1 + 9 + 9 - 6$

$14 = -1 + 9 + (9 - 6)!$

$15 = 1 \times 9 + (9 - 6)!$

$16 = 19 - 9 + 6$

$17 = -1 + 9 + \sqrt{9} + 6$

$18 = 1 \times 9 + \sqrt{9} + 6$

$19 = 1 + \sqrt{9} + 9 + 6$

$20 = -1 + \sqrt{9} + \sqrt{9} \times 6$

$21 = 1 \times \sqrt{9} + \sqrt{9} \times 6$

$22 = 19 + 9 - 6$

$23 = -1 + 9 + 9 + 6$

$24 = 1 \times 9 + 9 + 6$

$25 = 1 + 9 + 9 + 6$

$26 = -1 + \sqrt{9} \times (\sqrt{9} + 6)$

$27 = (1 + \sqrt{9})! + 9 - 6$

$28 = 1 + \sqrt{9} \times (\sqrt{9} + 6)$

$29 = -1 + (\sqrt{9})! \times (\sqrt{9})! - 6$

$30 = (1 + 9) \times (9 - 6)$

$31 = 1 + (\sqrt{9})! \times (\sqrt{9})! - 6$

$32 = -1 + 9 \times \sqrt{9} + 6$

$33 = 1 \times 9 \times \sqrt{9} + 6$

$34 = 19 + 9 + 6$

$35 = -19 + 9 \times 6$

$36 = 1 \times (\sqrt{9} + \sqrt{9}) \times 6$

$37 = 1 + (\sqrt{9} + \sqrt{9}) \times 6$

$38 = -1 + \sqrt{9} + (\sqrt{9})! \times 6$

$39 = (1 + \sqrt{9})! + 9 + 6$

$40 = 1 + \sqrt{9} + (\sqrt{9})! \times 6$

$41 = -1 + (\sqrt{9})! + (\sqrt{9})! \times 6$

$42 = (1 + \sqrt{9}) \times 9 + 6$

$43 = 1 + (\sqrt{9})! + (\sqrt{9})! \times 6$

$44 = -1 - 9 + 9 \times 6$

$45 = 1 \times (-9 + 9 \times 6)$

$46 = 1 - 9 + 9 \times 6$

$47 = -1 + 9 \times (\sqrt{9})! - 6$

$48 = 1 \times 9 \times (\sqrt{9})! - 6$

$49 = 1 + 9 \times (\sqrt{9})! - 6$

$50 = -1 - \sqrt{9} + 9 \times 6$

$51 = (-1) \times \sqrt{9} + 9 \times 6$

$52 = 1 - \sqrt{9} + 9 \times 6$

$53 = (-1) \times 9 \times (9 - 6)!$

$54 = 1 \times 9 \times (9 - 6)!$

$55 = 1 + 9 \times (9 - 6)!$

$56 = -1 + \sqrt{9} + 9 \times 6$

$57 = 19 \times (9 - 6)$

$58 = 1 + \sqrt{9} + 9 \times 6$

$59 = -1 + 9 \times (\sqrt{9})! + 6$

$60 = (19 - 9) \times 6$

$61 = 1 + 9 \times (\sqrt{9})! + 6$

$62 = -1 + 9 + 9 \times 6$

$63 = 1 \times 9 + 9 \times 6$

$64 = 1 + 9 + 9 \times 6$

$65 = 1 + [(\sqrt{9})! \div \sqrt{9}]^6$

$66 = (-1 + 9) \times 9 - 6$

$67 = 1 + (9 + \log_3 9) \times 6$

$68 = 1 + \sqrt{9} + (\log_3 9)^6$

$69 = [1 + (\sqrt{9})!] \times 9 + 6$

$70 = (1 + \sqrt{9})^{\sqrt{9}} + 6$

$71 = -1 + (\sqrt{9} + 9) \times 6$

$72 = (1 + \sqrt{9}) \times \sqrt{9} \times 6$

$73 = 19 + (9 \times 6)$

$74 = -1 + 9 \times 9 - 6$

$75 = 1 \times 9 \times 9 - 6$

$76 = 1 + 9 \times 9 - 6$

$77 = -19 + 96$

$78 = (-1 + 9) \times 9 + 6$

$79 = 1 + (\log_{\sqrt{3}} 9)! + 9 \times 6$

$80 = -1 + 9 \times (\sqrt{9} + 6)$

$81 = 1 \times 9 \times (\sqrt{9} + 6)$

$82 = 1 + 9 \times (\sqrt{9} + 6)$

$83 = -1 + 9! \div [(\sqrt{9})! \times 6!]$

$84 = (1 + 9) \times 9 - 6$

$85 = 1 + 9! \div [(\sqrt{9})! \times 6!]$

$86 = -1 + 9 \times 9 + 6$

$87 = 1 \times 9 \times 9 + 6$

$88 = 1 + 9 \times 9 + 6$

$89 = -1 + (\sqrt{9})! \times (9 + 6)$

$90 = 1 \times (\sqrt{9})! \times (9 + 6)$

$91 = 1 + (\sqrt{9})! \times (9 + 6)$

$92 = -1 + 99 - 6$

$93 = 1 \times 99 - 6$

$94 = 1 + 99 - 6$

$95 = (-1)^9 + 96$

$96 = 1^9 \times 96$

$97 = 1^9 + 96$

$98 = -1 + \sqrt{9} + 96$

$99 = 1 \times \sqrt{9} + 96$

$100 = 1 + \sqrt{9} + 96$

Editor's Note: Note that expressions for 67, 68 and 79 as appeared above are not satisfactory. Can you figure out why?