Problem 3

Let $[x]$ denote the largest integer which does not exceed x. For example $[3.21] = 3$. For each positive integer n, define $a_n = n + [\frac{1997}{n}]$. Find the smallest number in the sequence $\{a_1, a_2, a_3, \ldots\}$.

Prize

One $100 book voucher

Problem 4

There are 2998 points inside a circle which has area 1 unit. Prove that it is possible to choose three points among them such that the triangle formed by using these three points as vertices has an area less than $\frac{1}{1998}$.

Prize

One $100 book voucher

CONTEST

1. *Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.*
2. *To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.*
3. *Solutions should be typed and sent to:*

   The Editor, Mathematics Medley,
   c/o Department of Mathematics
   National University of Singapore
   Kent Ridge, Singapore 119260

   and should arrive before 31 January 1998.
4. *The Editor's decision will be final and no correspondence will be entertained.*

Problems ? Corner

Solutions

Solutions to the problems in Volume 24 No. 1 March 1997

Problem 1

Let a and b be two non-zero integers such that $|a| \le 400$ and $|b| \le 399$. Prove that

$$|a\sqrt{3} + b\sqrt{8}| > 1/1997.$$

Solution to Problem 1

by the Editor

Since a and b are non-zero integers we have $a\sqrt{3} \pm b\sqrt{8} \neq 0$. Therefore

$$|(a\sqrt{3} + b\sqrt{8})(a\sqrt{3} - b\sqrt{8})| = |3a^2 - 8b^2|$$

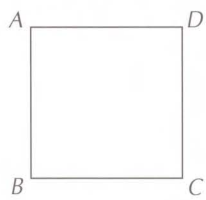
is a positive integer and hence $|3a^2 - 8b^2| \ge 1$.

Therefore $|a\sqrt{3} + b\sqrt{8}| \ge \dfrac{1}{|a\sqrt{3} - b\sqrt{8}|} > \dfrac{1}{2|a| + 3|b|} \ge \dfrac{1}{800 + 1197} = \dfrac{1}{1997}$.

Editor's note: No solutions from students were received for this problem.

Problem 2

In the figure $ABCD$ is a square. Given that there is a point P on the same plane such that $PA = 3$, $PB = 7$ and $PD = 5$. Find the area of $ABCD$ when



(i) P is inside $ABCD$;
(ii) P is outside $ABCD$.

Solution to Problem 2

Composite solution by Eric Fang, RJC, Class 2S01C;

Woo Sin Ai, Nanyang Junior College, Class 2CT30 and Woo Liyi, Woodland Secondary School, Class 4C.

The answers are (i) 58 and (ii) 16.

Introduce co-ordinates so that B is at $(0, 0)$, A is at $(0, a)$, C is at $(a, 0)$ and D is at

(a, a) where a is the length of the square.

Let (x, y) be the co-ordinates of P.

Then we have the system of equations

$$\begin{cases} (a-x)^2 + (a-y)^2 = 25 \\ x^2 + (a-y)^2 = 9 \\ x^2 + y^2 = 49 \end{cases}$$

and from the first two of these equations we find that $x = \dfrac{a^2 - 16}{2a}$ and $y = \dfrac{a^2 + 40}{2a}$.

Substituting these into the third equation we obtain the equation

$$a^4 - 74a^2 + 928 = 0.$$

Solving this equation we find that $a^2 = 58$ or $a^2 = 16$.

When $a^2 = 58$ we have $(x, y) = (\frac{21}{\sqrt{58}}, \frac{49}{\sqrt{58}})$ which shows that P is inside the square and when $a^2 = 16$ we have

$(x, y) = (0, 7)$ which shows that P is outside the square.

Editor's note: The $100 prize money was shared as follows: Eric Fang received $50, Woo Liyi and Woo Sin Ai each received $25.