Area Analysis in Geometric Argument

by
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Introduction. Analytic or, more generally, computational derivations in geometry generally start with equations. Thus there is the need to generate equations relevant to the geometry at hand - equations which are nothing more than statements that two things are really the same. The sources of such equations in analytic geometry are usually slopes and lengths. When we need to deal with angles, trigonometry is introduced. If, for example, we wish to obtain the law of cosines, it is customary to compute the square of the length of an altitude of a triangle in two ways. We then declare that the results are the same, thus giving rise to an equation which may be restated as the desired law.

On the other hand, when we wish to derive the law of sines, we usually compute areas first. We write the area of a triangle using the sines of two different angles. We then obtain an equation by setting the two expressions for the area to be equal and, from this equation, the law of sines follows.

Another familiar use of area appears in the result that the radius (r) of the circle inscribed in a triangle is given by A/s where A is the area and s the semiperimeter of the triangle. However, we could compute with slopes and distances. We could write equations for the bisectors of the angles of the triangle, solve the equations simulataneously for the coordinates of the incenter, and compute the distance from the incenter to a side of the triangle. No one would think that to be a good plan. Nevertheless, it seems to us that the use of area at the start of geometric arguments is not natural for most students or their teachers. They usually start with slopes and lengths and are often reluctant to abandon these ideas when they prove unproductive.

We like to use the term “area analysis” to refer to geometrical derivations which are based upon the computation of areas. In this paper we present a collection of eight problems in area analysis. Our hope is twofold. First, we hope that readers will enjoy the problems. Second, we hope that readers will be convinced that area should join slope and length as standard points of departure in computational derivations and proofs.

We also note that, in the work to follow, $[ABC]$ will represent the area of $\triangle ABC$.

Problem 1. In $\triangle ABC$, the internal bisector of $\angle A$ meets $\overline{BC}$ at point D, and the bisector of the external angle at A (as shown in Figure 1) meets the extension of $\overline{BC}$ and E.

Show that

$$AD = \sqrt{bc\left(1 - \frac{a^2}{(b+c)^2}\right)} \text{ and } AE = \sqrt{bc\left(\frac{a^2}{(c-b)^2} - 1\right)}$$

where $a = BC$, $b = AC$ and $c = AB$.

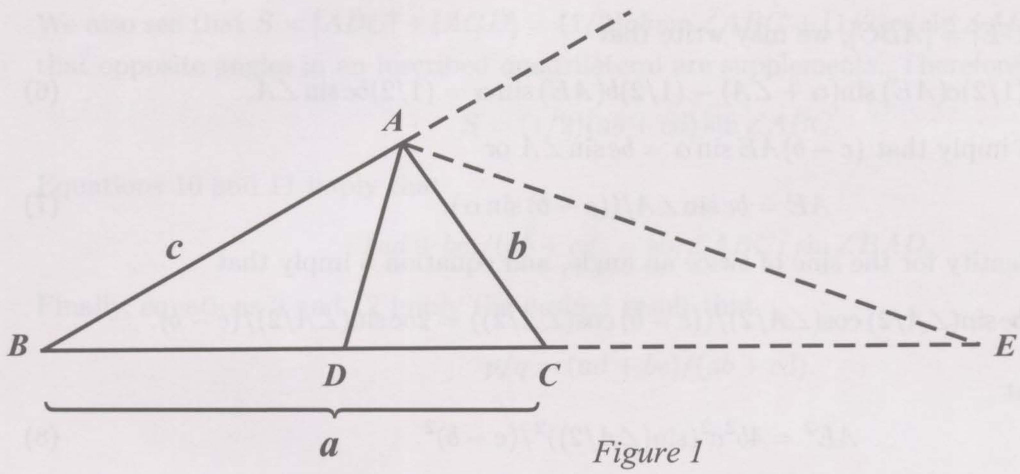

Figure 1

Solution. Since $\vec{AD}$ bisects $\angle A$, we may write that

$$[ABD] = (1/2)c(AD)\sin(\angle A/2)$$

and that

$$[ACD] = (1/2)b(AD)\sin(\angle A/2).$$

Since $[ABC] = [ABD] + [ACD]$, we may also write that

$$[ABC] = (1/2)(b+c)\sin(\angle A/2). \quad (1)$$

Furthermore, we see that

$$[ABC] = (1/2)bc\sin\angle A. \quad (2)$$

It follows from equations 1 and 2 that

$$AD = bc\sin\angle A/((b+c)\sin(\angle A/2)) = 2bc\sin(\angle A/2)\cos(\angle A/2)/((b+c)\sin(\angle A/2))$$

which implies that

$$AD = 2bc\cos(\angle A/2)/(b+c). \quad (3)$$

Next, we square both sides of equation 3, use the trigonometric identity $(\cos(\angle A/2))^2 = (1 + \cos\angle A)/2$, and recall the law of cosines to obtain

$$AD^2 = 4b^2c^2(\cos(\angle A/2))^2/(b+c)^2 = 2b^2c^2(\cos\angle A + 1)/(b+c)^2 = bc(b^2+c^2-a^2+2bc)/(b+c)^2.$$

Direct simplification yields

$$AD^2 = bc\left(1 - \frac{a^2}{(b+c)^2}\right).$$

This equation implies that

$$AD = \sqrt{bc\left(1 - \frac{a^2}{(b+c)^2}\right)}$$

as desired.

To obtain our expression for AE, we begin by letting $\angle CAE = \alpha$. Since $\angle A + 2\alpha = 180°$, we may write that

$$\alpha = 180° - (\alpha + \angle A) \quad (4)$$

and

$$\alpha = 90° - \angle A. \quad (5)$$

Since $[ABE] - [ACE] = [ABC]$, we may write that

$$(1/2)c(AE)\sin(\alpha + \angle A) - (1/2)b(AE)\sin\alpha = (1/2)bc\sin\angle A. \tag{6}$$

Equations 4 and 6 imply that $(c - b)AE\sin\alpha = bc\sin\angle A$ or

$$AE = bc\sin\angle A/((c - b)\sin\alpha). \tag{7}$$

Equation 7, the identity for the sine of twice an angle, and equation 5 imply that

$$AE = 2bc\sin(\angle A/2)\cos(\angle A/2)/((c - b)\cos(\angle A/2)) = 2bc\sin(\angle A/2)/(c - b).$$

It then follows that

$$AE^2 = 4b^2c^2(\sin(\angle A/2))^2/(c - b)^2. \tag{8}$$

The trigonometric identity $\sin(\angle A/2))^2 = (1-\cos\angle A)/2$ and the law of cosines applied to equation 8 yield

$$AE^2 = 2b^2c^2(1 - \cos A)/(c - b)^2 = bc(a^2 - (c - b)^2)/(c - b)^2 = bc(a^2/(c - b)^2 - 1).$$

Our second result follows immediately:

$$AE = \sqrt{bc(a^2/(c - b)^2 - 1)}.$$

Problem 2. Quadrilateral $ABCD$ is inscribed in a circle as shown in Figure 2. Let us adopt the notation that $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = p$ and $BD = q$.

Prove that $p/q = (ad + bc)/(ab + cd)$.

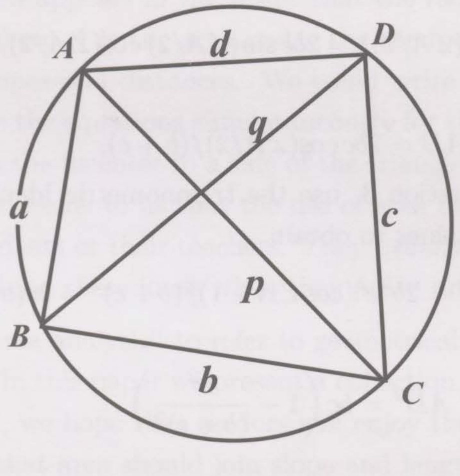

Figure 2

Solution. Let the length of the radius of the circle be R. The extended law of sines applied to $\triangle ABC$ inscribed in the circle implies that $p = 2R\sin\angle ABC$. Likewise, from $\triangle BAD$, we have $q = 2R\sin\angle BAD$.

Therefore

$$p/q = \sin\angle ABC/\sin\angle BAD. \tag{9}$$

Now let $[ABCD] = S$. Then $S = [ABD] + [CBD] = (1/2)ad\sin\angle BAD + (1/2)bc\sin\angle BCD$. Since $\angle BAD$ and $\angle BCD$ are supplements, we may write that

$$S = (1/2)(ad + bc)\sin\angle BAD. \tag{10}$$

We also see that $S = [ABC] + [ACD] = (1/2)ab \sin \angle ABC + (1/2)cd \sin \angle ADC$. Again, we note that opposite angles in an inscribed quadrilateral are supplements. Therefore

$$S = (1/2)(ab + cd) \sin \angle ABC. \tag{11}$$

Equations 10 and 11 imply that

$$(ad + bc)/(ab + cd) = \sin \angle ABC / \sin \angle BAD. \tag{12}$$

Finally, equations 9 and 12 imply the desired result that

$$p/q = (ad + bc)/(ab + cd).$$

Problem 3. Square $ABCD$ is inscribed in a circle as shown in Figure 3.1. Point P is on the minor arc $\widehat{AB}$, and chords $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ and $\overline{PD}$ are drawn.

Shown that

$$(PC)(PD) = (PA)(PB) + (PA)(PD) + (PB)(PC).$$

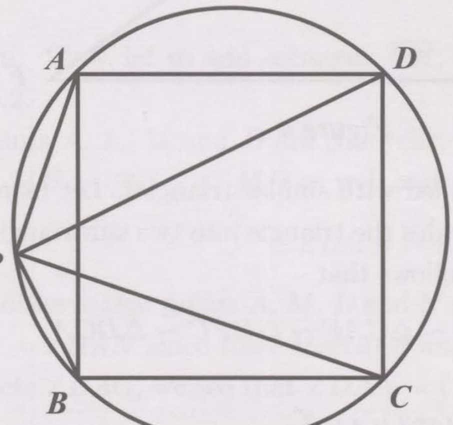

Figure 3.1

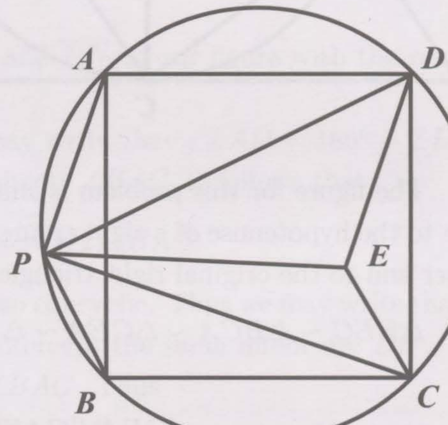

Figure 3.2

Solution. Draw the line through P which is parallel to $\overline{AD}$ and $\overline{BC}$. Find the point E inside the square and on this line for which $PE = AD = BC$. Then draw $\overline{PE}$, $\overline{DE}$ and $\overline{CE}$ as shown in Figure 3.2.

Since $\overline{PE} \| \overline{AD} \| \overline{BC}$ and $PE = AD = BC$, both $ADEP$ and $PECB$ are parallelograms. Therefore, $\triangle APD \cong \triangle EDP$, $\triangle PBC \cong \triangle CEP$, and $\triangle APB \cong \triangle DEC$. It is clear that

$$[PDC] = [EDP] + [CEP] + [DEC] = [APD] + [PBC] + [APB]. \tag{13}$$

Since each side of the square subtends a 90° arc, $\angle CPD = \angle APD = \angle BPC = 45°$ and $\angle APB = 180° - 45° = 135°$. Then

$$[PDC] = (1/2)(PC)(PD) \sin 45°, \quad [APD] = (1/2)(PA)(PD) \sin 45°,$$
$$[PBC] = (1/2)(PB)(PC) \sin 45°, \quad [APB] = (1/2)(PA)(PB) \sin 45°. \tag{14}$$

Substituting the areas of the triangles given by equations 14 into equation 13 yields

$$(PC)(PD) = (PA)(PB) + (PA)(PD) + (PB)(PC)$$

as desired.

Problem 4. Figure 4 shows rectangle $ABCD$ inscribed in a circle. Diagonal $\overline{AC}$ is, of course a diameter of the circle. The line tangent to the circle at C has been drawn, and sides $\overline{AB}$ and $\overline{AD}$ of the rectangle have been extended to intersect the tangent line at E and F, respectively.

Show that

$$BE/DF = (AE)^3/(AF)^3.$$

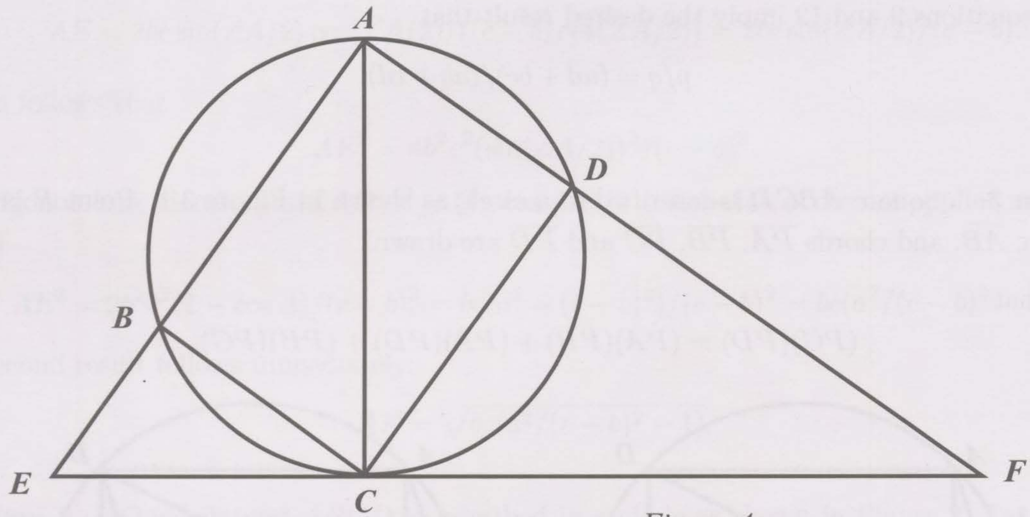

Figure 4

Solution. The figure for this problem is simply filled with similar triangles. Let us recall that the altitude to the hypotenuse of a right triangle divides the triangle into two subtriangles similar to each other and to the original right triangle. It follows that

$$\triangle BEC \sim \triangle BCA \sim \triangle CEA \sim \triangle AEF \sim \triangle CAF \sim \triangle DFC \sim \triangle DCA.$$

Then

$$[CEA]/[CAF] = (AE)^2/(AF)^2. \tag{15}$$

It is also true that

$$[CEA]/[CAF] = (EC)(AC)/((CF)(AC)) = EC/CF. \tag{16}$$

Equations 15 and 16 imply that

$$(AE)^2/(AF)^2 = EC/CF.$$

Squaring both sides of the last equation yields

$$AE^4/AF^4 = EC^2/CF^2. \tag{17}$$

Since $EC/BE = AE/EC$ and $CF/DF = AF/CF$ imply that $EC^2 = (BE)(AE)$ and $CF^2 = (DF)(AF)$, we see that equation 17 may be rewritten as

$$(AE)^4/(AF)^4 = (BE)(AE)/((DF)(AF)).$$

Thus we are able to conclude that

$$BE/DF = (AE)^3/(AF)^3$$

as required.

Problem 5. Suppose that $\triangle ABC$ represents any acute, non-isosceles triangle. The triangle is shown in Figure 5.1 with angle bisector $\overline{AD}$ and median $\overline{AM}$. The circle determined by points A, M and D is shown to intersect side $\overline{AB}$ at L and side $\overline{AC}$ at N.

Prove that $BL = NC$.

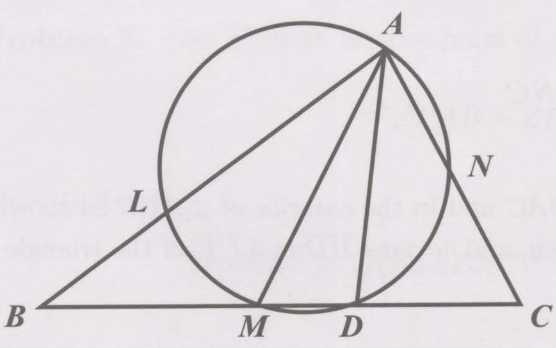

Figure 5.1

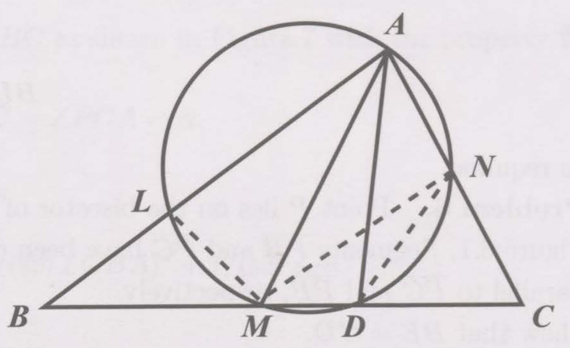

Figure 5.2

Solution. First, let us add segments $\overline{LM}$, $\overline{NM}$ and $\overline{DN}$ to our figure with the result shown as Figure 5.2.
Since points A, L, M and D are concyclic, we may write that $\angle LAD = 180° - \angle LMD$. Then, since $\angle LMB = 180° - \angle LMD$ as well and $\vec{AD}$ bisects $\angle BAC$, it follows that

$$\angle LMB = \angle LAD = (1/2)\angle BAC. \tag{18}$$

We now observe that points A, M, D and N are also concyclic. Thus we may write that $\angle CMN = \angle DMN = \angle DAN$ since these inscribed angles intercept the same minor arc $\widehat{DN}$. Again, since $\vec{AD}$ bisects $\angle BAC$, we see that $\angle DAN = (1/2)\angle BAC$. Thus

$$\angle CMN = (1/2)\angle BAC. \tag{19}$$

Let us next compare the areas of triangles BML and CMN. We write that

$$[BML]/[CMN] = (1/2)(BM)(ML)\sin\angle LMB/[(1/2)(CM)(MN)\sin\angle(CMN)].$$

Equations 18 and 19 and the fact that $CM = BM$ imply that

$$[BML]/[CMN] = ML/MN. \tag{20}$$

We now consider a third set of concyclic points A, L, M and N. We see that

$$\angle BLM = 180° - \angle ALM = \angle ANL = 180° - \angle MNC.$$

Thus

$$\angle BLM = 180° - \angle MNC. \tag{21}$$

By a comparison of the areas of a second pair of triangles, BML and CMN in this case, we have

$$[BML]/[CMN] = (1/2)(BL)(ML)\sin\angle BLM/((1/2)(MN)(NC)\sin\angle MNC).$$

Equation 21 implies that

$$[BML]/[CMN] = (BL)((ML)/[(MN)(NC)]). \quad (22)$$

Equations 20 and 22 together imply that

$$ML/MN = (BL)(ML)/((MN)(NC))$$

or

$$BL = NC$$

as required.

Problem 6. Point P lies on the bisector of $\angle BAC$ and in the exterior of $\triangle ABC$ as shown in Figure 6.1. Segments $\overline{PB}$ and $\overline{PC}$ have been drawn, and cevians $\overline{BD}$ and $\overline{CE}$ of the triangle are parallel to $\overline{PC}$ and $\overline{PB}$, respectively.

Show that $BE = CD$.

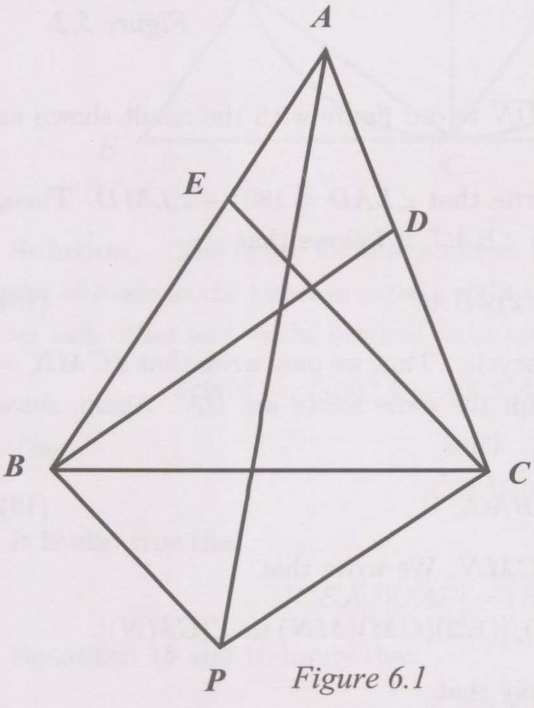

Figure 6.1

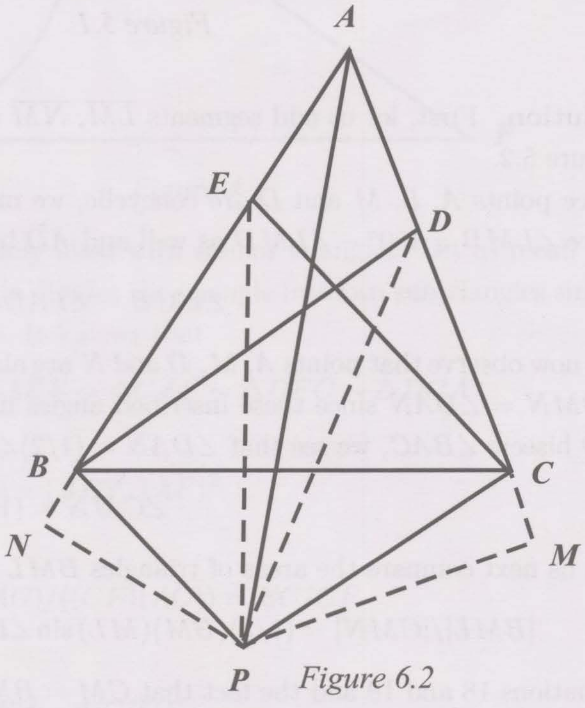

Figure 6.2

Solution. Let us draw $\overline{PM}$ perpendicular to the extension of $\overline{AC}$ at M and $\overline{PN}$ perpendicular to the extension of $\overline{AB}$ at N. Then let us draw $\overline{PD}$ and $\overline{PE}$. The results are shown in Figure 6.2.

We see immediately that $PN = AP \sin \angle BAP$ and $PM = AP \sin \angle CAP$. Since $\vec{AP}$ bisects $\angle BAC$, $\angle BAP = \angle CAP$. Therefore

$$PN = PM. \quad (23)$$

Also, $[PBE] = [PBC]$ since $\triangle PBE$ and $\triangle PBC$ share a common base $\overline{PB}$ and their vertices E and C lie together on a line parallel to that common base.

By the same sort of argument, $[PDC] = [PBC]$ since $\overline{BD} \| \overline{PC}$. Therefore,

$$[PBE] = [PDC]. \tag{24}$$

Since $[PBE] = (1/2)(BE)(PN)$ and $[PDC] = (1/2)(CD)(PM)$, we may conclude from equations 23 and 24 that $BE = CD$.

Problem 7. Let P be an interior point of $\triangle ABC$ as shown in Figure 7 with the property that

$$\angle PAB = \angle PBC = \angle PCA = \alpha.$$

Show that

$$1/(\sin\alpha)^2 = 1/(\sin\angle BAC)^2 + 1/(\sin\angle CBA)^2 + 1/(\sin\angle ACB)^2.$$

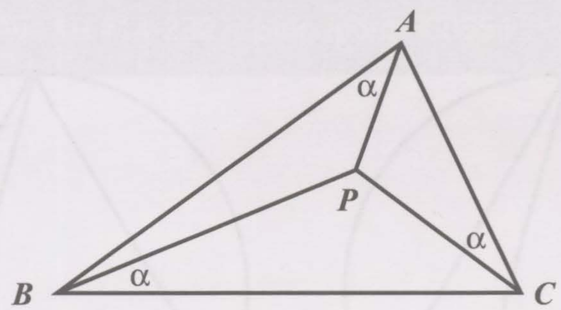

Figure 7

Solution. We begin by observing that $[ABC] = [PAB] + [PBC] + [PAC]$ from which it follows that

$$[PAB]/[ABC] + [PBC]/[ABC] + [PAC]/[ABC] = 1.$$

Therefore

$$(PA)(AB)\sin\angle\alpha/((AB)(AC)\sin\angle BAC) + (PB)(BC)\sin\alpha/((AB)(BC)\sin\angle CBA)$$
$$+(PC)(AC)\sin\alpha/((BC)(AC)\sin\angle ACB) = 1.$$

A bit of cancellation of common factors yields

$$PA\sin\angle\alpha/(AC\sin\angle BAC) + PB\sin\alpha/(AB\sin\angle CBA) + PC\sin\alpha/(BC\sin\angle ACB) = 1. \tag{25}$$

From $\triangle APC$,

$$\angle APC = 180° - (\alpha + \angle CAP) = 180° - \angle BAC.$$

Then from the law of sines,

$$AC/\sin\angle APC = AC/\sin\angle BAC = PA/\sin\alpha.$$

Therefore

$$PA/AC = \sin\angle\alpha/\sin\angle BAC.$$

Similarly, $PB/AB = \sin\angle\alpha/\sin\angle CBA$ and $PC/BC = \sin\alpha/\sin\angle ACB$. Substituting these ratios into equation 25, we obtain

$$(\sin\alpha)^2/(\sin\angle BAC)^2 + (\sin\angle\alpha)^2/(\sin\angle CBA)^2 + (\sin\alpha)^2/(\sin\angle ACB)^2 = 1$$

or

$$1/(\sin\angle BAC)^2 + 1/(\sin\angle CBA)^2 + 1/(\sin\angle ACB)^2 = 1/(\sin\alpha)^2$$

as desired.

Problem 8. Equilateral triangle ABC is inscribed in a circle as shown in Figure 8.1. Point P belongs to $120°$ arc $\widehat{BC}$ and chords $\overline{AP}$ and $\overline{BP}$ have been drawn.

Show that $PA = PB + PC$.

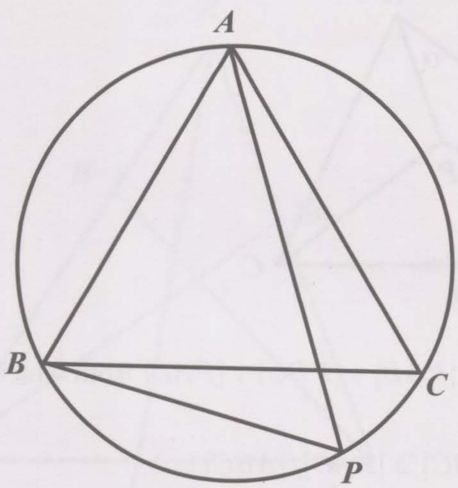

Figure 8.1

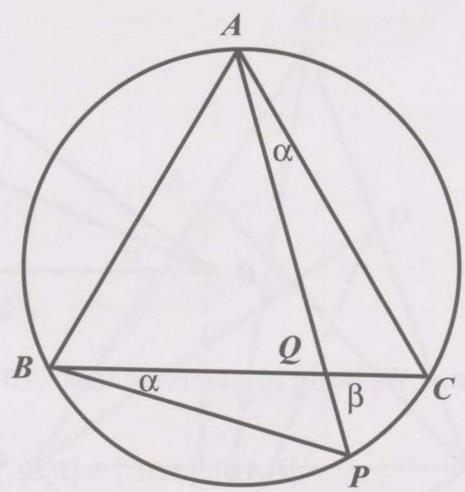

Figure 8.2

Solution. Let each side of $\triangle ABC$ have length denoted by a and let the measure of $\angle CAP = \alpha$. Then $\angle CBP = \alpha$ as well since $\angle CBP$ and $\angle CAP$ both intercept arc $\widehat{PC}$. Also, let Q denote the point at which $\overline{AP}$ and $\overline{BC}$ intersect and let $\angle CQP = \beta$. We display this notation in Figure 8.2. We see from our last figure that, as an exterior angle of $\triangle QAC$,

$$\beta = \alpha + \angle ACB = \alpha + 60°.$$

Next, we observe that

$$[ABPC] = (1/2)(PA)a\sin\beta = (1/2)(PA)a\sin(\alpha + 60°). \tag{26}$$

Also,

$$[ABP] = (1/2)(PB)a\sin\angle ABP = (1/2)(PB)a\sin(\alpha + 60°) \tag{27}$$

and

$$[ACP] = (1/2)(PC)a\sin\angle ACP = (1/2)(PC)a\sin(180° - ABP).$$

We note that the far right-hand part of equation follows since $\angle ACP$ and $\angle ABP$ are opposite angles in an inscribed quadrilateral; hence $\angle ACP = 180° - \angle ABP$. Thus

$$[ACP] = (1/2)(PC)a\sin\angle ABP = (1/2)(PC)a\sin(\alpha + 60°). \quad (28)$$

Since $[ABPC] = [ABP] + [ACP]$, we may infer from equations 26, 27 and 28 that

$$(1/2)(PA)a\sin(\alpha + 60°) = (1/2)(PB)a\sin(\alpha + 60°) + (1/2)(PC)a\sin(\alpha + 60°).$$

Therefore, $PA = PB + PC$ as desired.

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