Kings in Tournaments (2)

by
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Abstract

In [5], we have characterized those arcs e in a tournament T, such that the digraph $T - \{e\}$ obtained by deleting e from T contains a king. In this note, we characterize those pairs of arcs $\{e_1, e_2\}$, where $e_1 = (a, b)$ and $e_2 = (b, c)$ such that the digraph $T - \{e_1, e_2\}$ obtained by deleting these two arcs from T contains a king.

1. Tournaments

A *tournament* is a non-empty finite set of vertices in which every two vertices are joined by one and only one arrow (such an arrow is also called an *arc* or a *directed edge*).

Let T be a tournament and x, y be two vertices in T. If there is an arrow from x to y, we say that x **dominates** y or y is **dominated by** x (symbolically, $x \to y$). An arc from x to y is denoted by (x, y). The number of vertices dominated by x is the **out-degree** of x, and is denoted by $d^+(x)$. The number of vertices that dominate x is the **in-degree** of x, denoted by $d^-(x)$. The set of vertices dominated by x is the **out-set** of x, $O(x)$; and the remaining set of vertices that dominate x is the **in-set** of $x, I(x)$.

Let x be a vertex and A be any set of vertices not containing x in T. We write $A \to x$ to indicate that every vertex in A dominates x; and $x \to A$ to indicate that x dominates all the vertices in A. We write $A \Rightarrow x$ to indicate that at least one vertex in A dominates x; and $x \Rightarrow A$ to indicate that x dominates at least one vertex in A.

For any two vertices x, y in T, the **distance from** x **to** y, denoted by $d(x, y)$, is the *minimum* number of arrows one has to follow in order to travel from x to y. Clearly, $d(x, y) = 1$ if x dominates y; $d(x, y) \geq 2$ if x does not dominate y. Also, we write $d(x, y) = \infty$ if y is not reachable from x.

2. Kings in Tournaments

Let T be a tournament with $n \geq 2$ vertices. A vertex x in T is called the **emperor** if $d(x, y) = 1$ for any other vertex y in T; a vertex x in T is called a **king** if $d(x, y) \leq 2$ for any other vertex y in T.

Studying dominance relations in certain animal societies, the mathematical biologist Landau proved in [3] the following result:

Theorem 1. *In a tournament T, any vertex with the maximum score (out-degree) is always a king.*

Moon, a Canadian mathematician, proved in [4] the following:

Theorem 2. *In a tournament T, any non-emperor vertex v (i.e. v is dominated by some other vertex in T) is always dominated by a king.*

As a direct consequence of *Theorem 2*, we have:

Corollary 3. *No tournament contains exactly two kings.*

Thus, any tournament either contains exactly one king (the emperor) or at least three kings.

Let D be the resulting structure obtained from a tournament by deleting some arcs. A vertex x in D is called a **king** if $d(x, y) \leq 2$ for any other vertex y in D. In [5], we have proved the following result:

Theorem 4. *Let T be a tournament with at least three vertices and $e = (a, b)$ an arc in T. Let $D = T - \{e\}$. Then D contains at least one king if and only if $d^-(a) + d^-(b) \geq 1$ in D.*

3. The Main Result

Now let T be a tournament and e_1, e_2 be two arcs in T. Suppose e_1 and e_2 are deleted from T. Does the resulting structure still contain a king?

The objective of this note is to establish the following result.

Theorem 5. *Let T be a tournament with at least three vertices and $e_1 = (a, b), e_2 = (b, c)$ two arcs in T. Let $D = T - \{e_1, e_2\}$. Then D contains at least one king if and only if $d^-(a) + d^-(b) \geq 1$ and $d^-(b) + d^-(c) \geq 1$ in D, not counting (a, c) (i.e., $d^-(b) \geq 1$ OR $d^-(b) = 0$, $d^-(a) \geq 1$, and $d^-(c) \geq 1$ not counting (a, c).)*

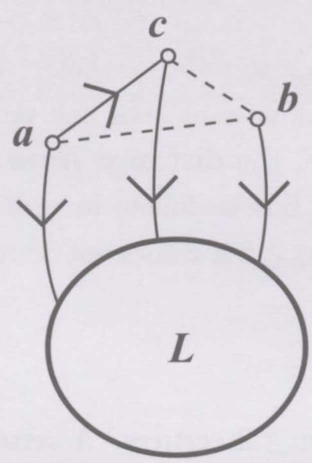

Figure 1

Note: The dotted lines show that the arcs (a, b) and (b, c) are deleted. L is the set of vertices excluding a, b and c in D. The dotted arrows indicate that the dominance relations are arbitrary and to be discussed.

Proof: [Necessity] Suppose on the contrary that in D, $d^-(a)+d^-(b) < 1$, i.e. $d^-(a)+d^-(b) = 0$, not counting (a,c). Then $d^-(a) = 0$ and $d^-(b) = 0$. In this case, $d(x,b) = \infty$ for every x in D and $d(b,a) \geq 3$. Thus D contains no kings. Similarly, if $d^-(b) + d^-(c) < 1$, then D contains no kings either.
[Sufficiency] Case (1) $d^-(b) \geq 1$ in D.

Let x be any vertex that dominates b, i.e. $x \to b$. If x is the emperor of $B(= D - \{b\})$, then x is the only king of D.

If x is not the emperor of B, then x is dominated by a king of B, by *Theorem 2*. Let this king be z. Clearly $d(z,b) = 2$, and thus z is a king of D (see Figure 2).

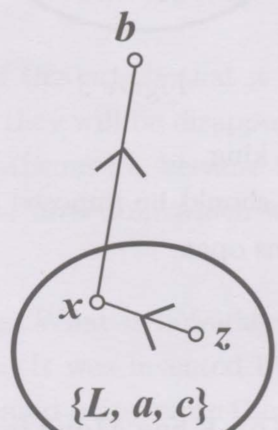

Figure 2

Case(2) $d^-(b) = 0$, $d^-(a) \geq 1$ and $d^-(c) \geq 1$ in D, not counting (a,c).

Then $b \to L$, and $d(x,b) = \infty$ for any x in D. Clearly, b is the only king of D since $b \to L \Rightarrow a$ and $b \to L \Rightarrow c$.

The proof of *Theorem 5* is thus complete. □

Now consider a more general problem. Let T be a tournament with at least four vertices and $e_1 = (a,b), e_2 = (c,d)$, where b and c may not be the same, be two arcs in T. Let $D = T - \{e_1, e_2\}$. Are similar conditions (i.e. $d^-(a)+d^-(b) \geq 1$ and $d^-(c)+d^-(d) \geq 1$ in D) sufficient to ensure the existence of a king of D? We can easily find one counter example: Suppose that in D, $d^-(a) = d^-(b) = d^-(c) = d^-(d) = 1$, and $a \to L$, $b \to L$, $c \to L$, $d \to L$ (see Figure 3).

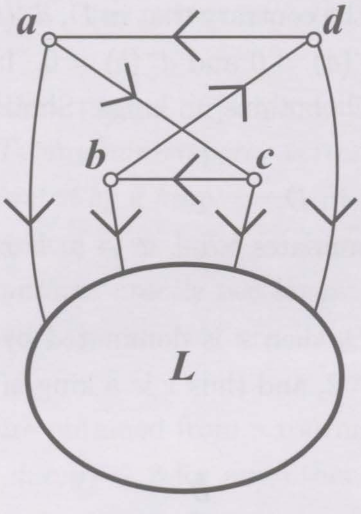

Figure 3

In this case, D does not contain a king.

Thus, what additional conditions should be imposed so that $D = T - \{e_1, e_2\}$ always contains a king? This problem remains open.

4. Acknowledgements

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References

[1] G.Chartrand and P. Zhang, "Introduction to Graph Theory", McGraw-Hill International Edition, 2005.

[2] K.M. Koh, "Digraphs", lecture notes, National University of Singapore.

[3] H.G. Landau, On dominance relations and the structure of animal societies III. The condition for a score structure. **Bull. Math. Biophys.**, 15(1953), 143 - 148.

[4] J.W. Moon, Solution to problem 463. **Math. Mag.**, 35(1962), 189.

[5] Y. Yu et al, "Kings in Tournaments", **Mathematical Medley**, 33(1)(2006), 28-32.