Mathematical Reasoning From O-Levels to A-Level

by
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1. Introduction

In the new A-Level syllabus document (see [1]), students are expected to reason mathematically, rather than to be burdened by complicated and mundane mathematical computations. In this note[1], some approaches to develop mathematical reasoning in A-Level mathematics students are discussed.

2. Students' Perceptions of A-Level Mathematics

My contact with A-Level students, including high-achieving Mathematics students, suggests that students in general have the following perception of A-Level Mathematics:

(a) A-Level Mathematics is "highly abstract" and "not derivable"

Generally, students feel that O-Level mathematics is more "concrete", and hence more easily understood, than A-Level mathematics. O-Level teachers tend to use various concrete approaches, including even hands-on activities, to teach difficult mathematics concepts, but not so much of A-Level teachers.

For example, students have a clear pictorial understanding of

$$(1+x)^2 = 1 + 2x + x^2.$$

After mastering the rules of algebraic expansion and without knowledge of the binomial theorem, students can derive the higher expansions

$$\begin{aligned}(1+x)^3 &= 1 + 3x + 3x^2 + x^3.\\ (1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4.\end{aligned}$$

In other words, the binomial theorem can be verified by direct algebraic expansion.

On the other hand, the binomial series covered in A-Level mathematics is the following:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots, \ -1 < x < 1,$$

where n is not a positive integer. This series does not appear to be derivable, nor even to make sense because: (i) it is an infinite series and not a polynomial, unlike its O-Level

[1] Presented to JC teachers at the Mathematics Teachers Conference 2006

counterpart, the binomial theorem, where n takes only positive integral values; (ii) there is an interval of convergence, unlike its O-Level counterpart, the binomial expansion, where positive integral values of n admit all values of x. Some students expressed that they were required only to memorize and apply the formula and not to concern themselves with its derivation.

(b) There are many topics in the syllabus which look alike but are taught separately.

For example, the binomial series as discussed above is usually taught in the Algebra section of the curriculum. In the calculus section, students are taught the Power series formula

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3}f^{(3)}(0) + \cdots$$

The above formula appears to be similar to the formula for binomial series, though they are not identical. To give another example, general curve sketching techniques and the sketching of trigonometric functions are taught in different sections of the curriculum, and could also be taught by different teachers.

(c) Some A-Level topics are "useless rituals"

Students see certain topics as useless rituals when they perceive that what they learnt in O-Levels is sufficient to solve the problems at hand; yet in the A-Levels teachers force them to acquire "weird knowledge" in order to deal with examination problems. For example, many students have already mastered "pattern-spotting" at O-Levels and are able to detect even rather complicated patterns. Thus the students do not see any purpose in going through the tedious steps of mathematical induction—except to score in the examinations.

(d) Some A-Level topics are "meaningless"

In the O-levels, students have been reminded repeatedly that square roots of negative numbers do not exist. Because of this, the entire chapter dealing with complex numbers does not make sense. While the computation involving complex numbers is generally not difficult for the average student, it is difficult for students to appreciate why complex numbers are useful.

3. Making "abstract" Mathematics more meaningful

Upon further thought, it can be seen that many A-Level Mathematics topics can be made more meaningful, without involving greater mathematical rigor. An effort can be made to help students understand mathematical results and the meaning of topics covered. For example, let us consider the topic of binomial series. When n is not a positive integer, we have

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots, \quad -1 < x < 1.$$

We suggest some activities to make the learning of this topic more interesting. (For more details, refer to [3].) These activities may be incorporated in the lectures or tutorial worksheets or both.

a. Without applying the formula for binomial series, and by simply comparing coefficients of "polynomials" (which the students have learnt in the O-Level mathematics), help the pupils to obtain the binomial expansion of $(1+x)^{-1}$, $(1+x)^{-2}$ and for other negative integral values of n. If one is more ambitious, he may even lead pupils to write down the expansion for some rational values of n, for example, $n = 1/2$. The basic tool that students need is how to compare coefficients of polynomials, which they have been taught in the O-Levels.

b. Guide pupils to observe that when n is not a positive integer, any (finite degree) polynomial cannot be the expansion of $(1+x)^n$.

c. It is not possible to introduce the concept of an interval of convergence for an infinite series at the A-Levels, in particular the interval of convergence for the binomial expansion of $(1+x)^n$. However, by making some observations with students, they can see that the expansion cannot be valid for $x \geq 1$ or $x \leq -1$.

d. One may also guide the students to observe that the above expansion is still valid when n is a positive integer; as in fact, the above formula is a generalization of the O-Level binomial formula and not separate from it. In such instances, the infinite series becomes a finite polynomial of degree n. It is not difficult for students to make such observations, and these would make learning of this topic more meaningful.

It is heartening to see that some junior colleges are making the teaching of power series more meaningful: If $f(x)$ is a function that is infinitely differentiable, with all its derivatives defined at 0, then it can be expressed as a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots .$$

By repeated differentiation, a_k can be computed as $\frac{f^{(k)}(0)}{k}$ where $f^{(k)}(0)$ denotes the value of the k-th derivative of f evaluated at 0.

4. Providing links across different Mathematics topics

Teachers could provide links across different topics in mathematics when they are visually similar. One good example discussed above is the relation between power series and binomial series. The power series of an infinitely differentiable function at 0 can be written as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{3} f^{(3)}(0) + \cdots ,$$

while the formula for the binomial series of $(1+x)^n$ is easily stated. In fact, the binomial series can be seen as a special case of the power series. The Maths C Special Paper question in June

88, Q1(a) is an example of an assessment question that tests this knowledge. Understanding mathematical concepts and the links between different topics is more important than mere mathematical computation.

In A-Level calculus, many more integration formulae and techniques are introduced, on top of the formulae and techniques that students learn in O-Levels. It is possible that students are overwhelmed by the volume of formulae and techniques that they miss out on understanding the essence of calculus. For example, the fact that definite integrals can be interpreted as areas or volumes of revolution is important and should be emphasized. Such a concept allows us to evaluate many definite integrals without tedious integration techniques. For example, the definite integral $\int_0^1 \sqrt{1-x^2}dx$ can be interpreted as the area of a quadrant of a unit circle, hence its value is $\pi/4$. For $0 < a < 1$, the integral $\int_0^a \sqrt{1-x^2}dx$ can also be interpreted as area under the graph. By using formulae for the area of triangles and the area of a sector, and without using integration techniques, it can be shown that

$$\int_0^a \sqrt{1-x^2}dx = \frac{1}{2}\sin^{-1} a + \frac{1}{2}a\sqrt{1-a^2}.$$

As another example, the integral $\int_0^1 \pi x^2 dx$ can be seen as the volume of a cone with unit height and base radius, and hence it evaluates to $\int_0^1 \pi x^2 dx = \pi/3$. By dividing by the constant π, we have $\int_0^1 x^2 dx = 1/3$. Teachers should pay more emphasis on the fundamental principles rather than overload the students with the voluminous techniques and formulae.

5. Students' understanding pattern generalization as mathematical "proof"

a) Pattern gazing and generalization as "proof": The current secondary school curriculum does not focus much on mathematical proofs. However, students are familiar with "pattern-spotting" or "generalizing" which they have practiced from their lower secondary days. After observing the first few values $f(1), f(2), f(3), \ldots$, students are good at formulating a general formula for $f(n)$.

For an example, consider the well-known Tower of Hanoi problem. The problem consists of a number of discs of different sizes and three poles. Initially, all the discs are on one pole, with the discs arranged from smallest (top) to biggest (bottom). The objective is to move all the discs from the first pole to the second pole, while obeying the rule that a bigger disc cannot be on top of a smaller disc. What is the minimum number of moves required to accomplish this objective? If the tower has one disc, only one step is needed to move this disc from the first pole to the second pole. If the tower has two discs, it takes a minimum of three steps to move the entire set of two discs from the first pole to the second. If the tower has three discs, it takes a minimum of seven steps to move the entire set from the first pole to the second. The students may continue to try what happens when there are four or five discs. Tabulating the data in a table, we have

No. of discs	Minimum no. of steps
1	1
2	3
3	7
4	15
5	31

By observing the data, most students would be able to postulate that for n discs, it takes a minimum of $2^n - 1$ steps to move all the n discs from the first pole to the second, while observing the rule that no bigger disc can be on top of a smaller disc. However, students should be reminded that this is only a conjecture and not a mathematical proof. If this generalizing of the data is seen as a "mathematical proof", then naturally students will view the steps of mathematical induction taught at A-Levels as redundant! Teachers must be able to demonstrate to students by using examples that sometimes the first few terms of a number sequence will suggest a certain pattern which might not be correct. The following example illustrates that pattern generalizing may lead to a "wrong formula". One of the SMO (Junior) questions of 2004 is as follows: N points are selected on the circumference of a circle and every two points are joined. What is the maximum number of regions formed inside the circle by all the chords thus formed?

No. of points N	No. of maximum possible regions
1	1
2	2
3	4
4	8
5	16

Based on pattern generalizing, one will erroneously derive that for N points, the maximum possible number of regions is 2^{N-1}. However, by drawing the case when $N = 6$, we can see that there is a maximum of 31 possible regions and not 32, and for $N = 7$, there are only 57 regions and not 64. In fact, the true maximum number of regions for N points is given by

$$N + \frac{N(N-1)(N-2)(N-3)}{24} + \frac{(N-2)(N-1)}{2},$$

which can be obtained by combinatorial argument. One can refer to [2] to read about this problem. This example illustrates that pattern generalizing is not a foolproof way of finding a formula.

b) Inductive reasoning as mathematical "proof": Many students at the secondary school levels have been shown illustrations, especially of difficult concepts (see for example [4]), using computer tools or softwares. However, these are only illustrations and are by no means mathematical proofs that require step-by-step logical deduction.

6. Mathematical Induction

Mathematical induction is meaningful to students only if they understand that their generalization from patterns and inductive reasoning do not always give correct results. Teachers can help students to see that mathematical induction is not a series of meaningless rituals but are useful and meaningful steps.

a) More real problems should be used. While many past year exam questions involve students proving some rather artificial formula, mathematical induction can be introduced to students by allowing them to investigate problems, making conjecture on the formulae (building on their mastery of pattern gazing and generalization), and then using mathematical induction to prove their conjecture. As an illustration of such investigative problems, refer to the following past A-level special paper question:

Given a sequence of numbers $\{a_n\}$, where $a_1 = a_2 = 1$, and the subsequent terms a_n are obtained by adding up all the preceding terms. Obtain expressions for a_3, a_4, a_5 and a_6. Predict a formula for a_n, where $n \geq 1$. Prove your conjecture by mathematical induction.

b) Critical analysis involving induction steps. Many interesting fallacies from proof by induction can be discussed with the students. Students can be given the well-known incorrect "proofs" that (i) any set of n integers are all equal; and (ii) the number e is rational. The students can be asked to critically examine the steps and mistakes in the steps, thereby enhancing mathematical thinking in reading mathematical statements.

c) Use a variety of examples. It is unfortunate that mathematical induction is classified under summation of series in the new A-level syllabus. However, in order to demonstrate the essential idea of mathematical induction, one should use more varied induction problems besides sums of series; for example inequalities, recurrence relations (of which the Tower of Hanoi problem above is an instance), sequences and other types of problems should be included. It is common that students take the statement

$$\sum_{k=1}^{n+1} f(k) = \sum_{k=1}^{n} f(k) + f(n+1)$$

blindly as a ritual, rather than see it as something meaningful.

7. Making "meaningless" mathematics useful

As mentioned above, most students may not have difficulty in performing arithmetic operations involving complex numbers. One could bring to focus on the applications of complex numbers to other branches of elementary mathematics which the students are already familiar with.

a) Application to real numbers. For any two complex numbers z and w, we have the formula $|zw| = |z| \cdot |w|$, where $|x|$ is the modulus of the complex number x. When this formula is applied to integers, it says that the product of two integers which are sums of two square integers is again a sum of two square integers. Students can be led to see the connection between complex numbers and its relation to such results in number theory.

b) Application to geometry. As another example, the classical proof of Ptolemy's Theorem for cyclic quadrilaterals involves constructing lines and using similar triangles. However, complex numbers can offer an alternative proof to the theorem.

Teachers might amuse their students with such interesting applications of complex numbers to different areas of mathematics.

8. Conclusion

The above are some recommendations, which are by no means exhaustive, of teaching A-level mathematics with an emphasis on mathematical reasoning and mathematical sense. Such recommendations only involve a change of perspective in the teaching approaches and not necessarily the expenditure of more curriculum time.

References

[1] Singapore Ministry of Education, 2006 A-level Mathematics syllabuses, Curriculum Planning and Development Division, 2004.

[2] Chen C.C., Koh K.M., Principles and Techniques in Combinatorics, World Scientific, 1992.

[3] Toh T.L., On Teaching Binomial Series: More Meaning and Less Drill, International Journal of Mathematics Education in Science and Technology, Vol 34(1), P115 - 121, 2002.

[4] Toh T.L., On Using Geometer's Sketchpad to Teach Relative Velocity, Asia-Pacific Forum on Science Learning and Teaching, Vol 4(2), 2003.

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