Mathematical Medley Problems Corner

Volume 33 No. 2 December 2006

A. Prized Problems

Problem 1. (Book voucher up to \$150)

Find all positive integers a, b, c, d, all of which between 1 and 9 inclusive, such that

$$\frac{1333a + 130b}{20c + 332d} = 1.$$

Problem 2. (Book voucher up to \$150)

Find all positive integers a, b and c such that

$$\frac{a^2 + b^2}{3ab - 1} = c.$$

Proposed by Albert F.S. Wong, Temasek Polytechnic

B. Instruction

(1) Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

(2) To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

(3) Solutions should be sent to : **The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore117543** ; and should arrive before 31 October 2007.

(4) The Editor's decision will be final and no correspondence will be entertained.

C. Solutions to the problems of volume 32, No2, 2005

Problem 1

Find all the real solution pairs (x, y) that satisfy the system

$$\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}} = (x+3y)(3x+y) \qquad (1)$$

$$\frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{y}} = 2(y^2 - x^2) \qquad (2)$$

(Note: $\sqrt{x}$ denotes the nonnegative square root of the real (nonnegative) number x).

Proposed by Albert Wong, Temasek Polytechnic.

Solution to Problem 1. *By Chew Yi De - Nanyang Junior College.*

Let $a = \sqrt{x}$ and $b = \sqrt{y}$. The two equations become

$$\frac{1}{a} + \frac{1}{2b} = (a^2 + 3b^2)(3a^2 + b^2). \qquad (3)$$

$$\frac{1}{a} - \frac{1}{2b} = 2(b^4 - a^4). \qquad (4)$$

Adding two equations above, we obtain

$$\frac{2}{a} = (3a^4 + 10a^2b^2 + 3b^4) + (2b^4 - 2a^4).$$

Thus, we get

$$2 = 5a^5 + 10a^3b^2 + 5b^5. \qquad (5)$$

On the other hand, by subtracting (4) from (3), we obtain

$$\frac{1}{b} = (3a^4 + 10a^2b^2 + 3b^4) - (2b^4 - 2a^4).$$

Thus, we get

$$1 = 5a^4b + 10a^2b^3 + b^5. \qquad (6)$$

By adding (5) and (6), we obtain $3 = (a+b)^5$. Similarly, by subtracting (6) from (5), we obtain $1 = (a-b)^5$. It is now easy to deduce that

$$a = \frac{1+\sqrt[5]{3}}{2} \text{ and } b = \frac{\sqrt[5]{3}-1}{2}.$$

Consequently, we obtain

$$x = \left(\frac{1+\sqrt[5]{3}}{2}\right)^2 \text{ and } b = \left(\frac{\sqrt[5]{3}-1}{2}\right)^2.$$

Editor's note: Similar and complete solutions to the above were received from Dai Zhonghuan (Hwa Chong Institution) and Dominic Lee Jun (NUS High School)

Problem 2

(a) Prove that for any positive integer n,

$$\sum_{k=0}^{\lfloor n/4\rfloor} \binom{n}{4k} = 2^{n-2} + (\sqrt{2})^{n-2} \cos\frac{n\pi}{4}$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

(b) Prove that

$$\sum_{k=1}^{45} \tan^2(2k-1)^\circ = 4005.$$

(Note: Solution by direct algorithmic computation will not be accepted)

Solution to Problem 2. *By Chew Yi De (Nanyang Junior College) and Wang Tengyao (Hwa Chong Institution)*

Observe that

$$2^n = (1+1)^n - (1-1)^n = \sum_{k=0}^{n}\binom{n}{k} - \sum_{k=0}^{n}(-1)^n\binom{n}{k} = \sum_{k=0}^{\lfloor n/2\rfloor} 2\binom{n}{2k}.$$

Consider now the complex number $z = (1+i)^n$. By using De Movire's Theorem, the real part of z is $2^{n/2}\cos\frac{n\pi}{4}$. On the other hand, if we expand the expression $(1+i)^n$, we see that the real part is

$$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n}{4k} - \sum_{k=0}^{\lfloor (n-2)/4\rfloor}\binom{n}{4k+2}.$$

It follows that

$$2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k} + \sum_{k=0}^{\lfloor n/4\rfloor}\binom{n}{4k} - \sum_{k=0}^{\lfloor (n-2)/4\rfloor}\binom{n}{4k+2} = \sum_{k=0}^{\lfloor n/4\rfloor} 2\binom{n}{4k}.$$

(a) now follows from the above equation.

For (b), we will prove the following more general result:

$$\sum_{k=1}^{n}\tan^2(\frac{(2n-1)\pi}{4n})^c = n(2n-1).$$

From now on, we will drop the superscript c. By using De Movire's Theorem again, we see that

$$\cos(2nx) = \sum_{k=0}^{n}(-1)^k\binom{2n}{2k}\cos^{2n-2k}x\sin^{2k}x.$$

Dividing the above equation by $\cos^{2n} x$, we obtain

$$\frac{\cos(2nx)}{\cos^{2n} x} = \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} \tan^{2k} x.$$

Observe that the above expression is 0 if $\cos(2nx) = 0$ and $\cos x \neq 0$. In particular, we see that for any integer s, if $x = \frac{(2s-1)\pi}{4n}$, then

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{2k} \tan^{2k} x = 0.$$

However, if we replace $\tan^2 x$ by t, it means that for any integer s, $\tan^2(\frac{(2s-1)\pi}{4n})$ is a solution of the following polynomial equation:

$$0 = \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} t^k.$$

For convenience, we rewrite the above equation as follows:

$$t^n - \binom{2n}{2(n-1)} t^{n-1} + \sum_{k=0}^{n-2} (-1)^k \binom{2n}{2k} t^k = 0. \tag{7}$$

Note that $\tan^2(\frac{\pi}{4n}), \tan^2(\frac{3\pi}{4n}), \ldots, \tan^2(\frac{(2n-1)\pi}{4n})$ are all the n distinct roots. Therefore in view of (7), $n(2n-1)$ is equal to sum of all roots. In other words, we obtain

$$\sum_{k=1}^{n} \tan^2(\frac{(2n-1)\pi}{4n}) = n(2n-1).$$

Now when $n = 45$, $(\frac{(2n-1)\pi}{4n})^c = (2n-1)^\circ$. Thus, we obtain

$$\sum_{k=1}^{45} \tan^2(2k-1)^\circ = 4005.$$

Editor's note: We received fewer correct solutions for Problem 2. The first part of the solution is due to Chew Yi De and the second part is due to Wang Tengyao. A similar and complete solutions to the above was also received from Dominic Lee Jun (NUS High School)

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