Andrews's argument proves a theorem of Zeckendorf

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The famous Fibonacci numbers F_n are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}\,(n \geq 2)$. In 1939, Zeckendorff [2] discovered his theorem:

Theorem 1 (Zeckendorf's theorem) *Every positive integer can be uniquely represented as a sum of non-consecutive Fibonnacci numbers from* $F_2, F_3, F_4, \cdots$.

In his proof, Zeckendorf treated the existence and the uniqueness of the representation separately by induction. In 1969, Andrews [1] gave an elegant proof of the basis representation theorem:

Theorem 2 (The basis representation theorem) *Let* k *be an integer* ≥ 2. *Then, for any positive integer* n, *there exists a unique representation*

$$n = a_0 + a_1 k + a_2 k^2 + \cdots + a_r k^r,$$

where $a_r \neq 0$, *and where each* a_i *is nonnegative and less than* k.

Most proofs of the basis representation theorem in the literature use induction and treat the existence part and the uniqueness part separately. Andrews's proof seems to be the first one which produces the existence and uniqueness at the same time. We sketch Andrews's proof here: Let $b_k(n)$ denote the number of representations of n to the base k. If $n = a_sk^s + a_{s-1}k^{s-1} + a_{s-2}k^{s-2} + \cdots + a_rk^r$, where $a_s \neq 0$, $a_r \neq 0$, and each a_i is nonnegative and less than k, then $n-1 = a_sk^s + a_{s-1}k^{s-1} + a_{s-2}k^{s-2} + \cdots + (a_r-1)k^r + \sum_{i=0}^{r-1}(k-1)k^i$ is a representation for $n-1$ to the base k. Thus $b_k(n) \leq b_k(n-1)$. Iteration of this inequality gives $1 \leq b_k(k^n) \leq b_k(n) \leq b_k(1) = 1$. And the result follows.

It seems that Andrews's proof has not been paid much attention. The purpose of this note is to show that Zeckendorf's theorem follows easily from an argument similar to Andrews's sketched in the above. First, we recall some properties about Fibonacci numbers. It is readily verified by induction that $n \leq F_{n+1}$ for $n \geq 2$, that $F_n < F_{n+1}\,(n \geq 2)$, and that

$$\begin{cases} F_3 + F_5 + F_7 + \cdots + F_{2n-1} = F_{2n} - 1, \ (n \geq 2) \\ F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1, \ (n \geq 1). \end{cases} \tag{1}$$

Using (1) and Andrews's argument, we are now in a position to give an new proof of Zeckendorf's theorem.

Proof of Zeckendorf's theorem. Let $R(n)$ be the number of representations of a positive integer n as a sum of Fibonacci numbers of the form:

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \tag{2}$$

where $k_{j+1} - k_j \geq 2$, for $j = 1, 2, \cdots, r-1$, and $k_1 \geq 2$. Our goal is to show that $R(n) = 1$ for $n = 1, 2, 3, \cdots$. The representation (2) is called a Zeckendorf representation. Note that $R(1) = 1$. Let $n \geq 2$ and suppose that $n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}$ is a Zeckendorf representation. It follows from (1) that

$$\begin{aligned} n - 1 &= (F_{k_1} - 1) + F_{k_2} + F_{k_3} + \cdots + F_{k_r} \\ &= \begin{cases} F_{k_2} + F_{k_3} + \cdots + F_{k_r} & \text{if } k_1 = 2, \\ F_3 + F_5 + \cdots + F_{k_1-1} + F_{k_2} + F_{k_3} + \cdots + F_{k_r} & \text{if } k_1 > 2 \text{ and is even}, \\ F_2 + F_4 + \cdots + F_{k_1-1} + F_{k_2} + F_{k_3} + \cdots + F_{k_r} & \text{if } k_1 > 2 \text{ and is odd}, \end{cases} \end{aligned}$$

is also a Zeckendorf representation of $n-1$. Thus we have

$$R(n) \leq R(n-1), \qquad \text{for } n \geq 2. \tag{3}$$

Note that (3) is true even if n has no representation. Since $n \leq F_{n+1}$ for $n \geq 2$ and since $R(F_{n+1}) \geq 1$ for $n \geq 2$, it follows from (3) that

$$1 \leq R(F_{n+1}) \leq \cdots \leq R(n) \leq \cdots \leq R(2) \leq R(1) = 1, \qquad \text{for } n \geq 2.$$

Therefore $R(n) = 1$ for $n = 1, 2, 3, \cdots$. □

We close with the following generalization of the basis representation theorem:

Theorem 3 *Let $a_0, a_1, a_2, a_3, \cdots$ be an infinite sequence of positive integers $B_0 = 1$, and $B_i = (1+a_0)(1+a_1)\cdots(1+a_{i-1})$ for $i \geq 1$. Then, for any positive integer n, there exist unique nonnegative integers $d_0, d_1, d_2, d_3, \cdots$, such that*

$$n = d_0 + d_1B_1 + d_2B_2 + d_3B_3 + \cdots + d_rB_r,$$

where $d_r \neq 0$ and $0 \leq d_i \leq a_i$, for $i = 0, 1, 2, \cdots, r$.

The proof of Theorem 3 is exactly that of Zeckendorf's theorem which we presented in the above, except that the role of (1) is now replaced by the identity:

$$\begin{aligned} &(1+x_0)(1+x_1)(1+x_2)\cdots(1+x_n) \\ = \;& 1 + x_0 + (1+x_0)x_1 + (1+x_0)(1+x_1)x_2 + \cdots + (1+x_0)(1+x_1)\cdots(1+x_{n-1})x_n. \end{aligned}$$

We leave the details to the interested reader.

References

[1] George E.Andrews, On radix representation and the Euclidean algorithm, *Amer. Math. Monthly*, 76 (1969) 66-68.

[2] Edouard Zeckendorf, Repr´esentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bulletin de la Soci´et´e Royale des Sciences de Li`ege*, 41 (1972) 179-182.