# Cover-up Rule for Partial Fractions

- a simple explanation using limits

Decomposing a rational function $\frac{P(x)}{Q(x)}$ into its partial fractions is a useful technique when performing integration of rational functions. For instance to find the anti-derivative of

$$\frac{5x+7}{(x+1)(x+2)},$$

we need to decompose the above rational function into its partial fractions first. To this end, we assume the rational function is equal to

$$\frac{a}{x+1}+\frac{b}{x+2},$$

then

$$\frac{5x+7}{(x+1)(x+2)}=\frac{a}{x+1}+\frac{b}{x+2}.$$

Thus,

$$\frac{5x+7}{(x+1)(x+2)}=\frac{(x+2)\,a+(x+1)\,b}{(x+1)(x+2)}. \tag{1}$$

Comparing numerators,

$$5x+7=(x+2)\,a+(x+1)\,b \tag{2}$$

Comparing the coefficient of $x$ and constant term, we have the simultaneous equations

$$\begin{aligned} 5 &= a+b, \\ 7 &= 2a+b. \end{aligned}$$

Solving the equations we have.

$$a=2 \quad \text{and} \quad b=3.$$

Hence

$$\frac{5x+7}{(x+1)(x+2)}=\frac{2}{x+1}+\frac{3}{x+2}.$$

Thus

$$\int \frac{5x+7}{(x+1)(x+2)}dx=\int \frac{2}{x+1}dx+\int \frac{3}{x+2}dx.$$

**Cover-Up Rule**

Some textbooks introduce a short cut to determining the coefficients $a$ and $b$. This method is called the Covering-up (or cover-up) rule. This is particularly useful for partial fractions with simple linear denominators. For instance, to determine the coefficient of $\frac{1}{(x+1)}$ in the above example, we cover up the factor $(x+1)$ in the denominator and put $x=-1$, i.e.,

$$a=\left.\frac{5x+7}{(\blacksquare\blacksquare\blacksquare)\,(x+2)}\right|_{x=-1}=\frac{-5+7}{-1+2}=2.$$

Covering up $(x+2)$ and putting $x=-2$, we have

$$b=\left.\frac{5x+7}{(x+1)\,(\blacksquare\blacksquare\blacksquare)}\right|_{x=-2}=\frac{-10+7}{-2+1}=3.$$

A usual "proof" of this method is as follows. Consider equation (2),

$$5x+7=(x+2)\,a+(x+1)\,b.$$

To get rid of $b$, we let $x=-1$, then we have

$$5\,(-1)+7=(-1+2)\,a+(0)\,b$$

thus

$$a=\frac{5\,(-1)+7}{(-1+2)},$$

which is the "covering-up" of the factor $(x+1)$ :

$$a=\left.\frac{5x+7}{(\blacksquare\blacksquare\blacksquare)\,(x+2)}\right|_{x=-1}$$

Similarly, letting $x=-2$ we obtain $b=3$. However, **this explanation is technically not correct.** We note that equation (2) is in fact comparing numerators of equation (1), which is **valid only when the denominator is nonzero**. As a result, equation (2) holds only when $x\neq-1$ and $x\neq-2$. Equation (2) should be stated as follows:

$$5x+7=(x+2)\,a+(x+1)\,b,\;\; x\neq-2,-1. \qquad \text{(Corrected (2))}$$

Therefore we may not substitute $x=-1$ or $x=-2$ into (2).

Nonetheless, (2) is valid for any number $x$ as long as $x\neq-1,-2$. Recall that the purpose of covering-up is to eliminate one the unknown coefficient. Instead of substituting $x=-1$ (which is not allowed), to eliminate $b$, we can subsitute $x$ closed to $-1$, but not equal to $-1$, e.g., we take $x=-0.9$, or $x=-0.999$, or $x=-0.9999999999$, etc., so that the "proportion" of $b$ diminishes as $x$ gets closer to $-1$.

**Limit of a function**

The process of evaluting a function $f\,(x)$ by substituting $x$ arbitarily closed to number $c$ (without taking $x=c$) is called finding the limits. For instance, if $f(x) = \dfrac{x^2-1}{x-1}$, and we want to find the limit when $x \to 1$, we need to substitute $x$ by a number near 1 (but not 1). For instance, $f(1.1) = \frac{1.21-1}{1.1-1} = 2.1$, $f(1.01) = \frac{1.0201-1}{1.01-1} = 2.01...$, also $f(0.9) = 1.9$, $f(0.99) = 1.99$, ... We see that as $x \to 1$, $f(x) \to 2$. In this case, we say that

$$\lim_{x \to 1} f(x) = 2.$$

Coming back to the cover-up rule, since Equation (2) is valid for all $x \neq -2, -1$, and $\lim_{x \to -1} (x+1) = 0$, to eliminate $b$,

we take limits as $x$ tends to $-1$ on both sides of (2),

$$\lim_{x \to -1} [5x+7] = \lim_{x \to -1} [(x+2)a] + \lim_{x \to -1} [(x+1)b]$$

thus

$$(5(-1)+7) = ((-1)+2)a$$

or

$$a = \frac{(5(-1)+7)}{((-1)+2)}.$$

which this the required "covering-up" of $(x+1)$.