What is Implication?

Let S be the set of natural numbers up to 40: $\{1, 2, 3, \ldots, 39, 40\}$. Classify every element of S two ways, according to whether it is divisible by 4, and whether it is divisible by 2.

	Divisible by 2	Not divisible by 2
Divisible by 4	$4, 8, 12, \ldots, 40$	
Not divisible by 4	$2, 6, 10, \ldots, 38$	$1, 3, 5, \ldots, 39$

Table 1: Classification of S via divisibility by 4 (rows) and by 2 (columns).

The empty cell means S contains no number which is divisible by 4 but not 2. I.e., if an element of S is divisible by 4, then it is divisible by 2. Equivalently,

On S, divisibility by 4 **implies** *divisibility by 2.*

The general definition of implication is as follows. Let Σ be a set, and let A and B be statements applicable to every element of Σ. Imagine placing every element in the correct cell in the Table 2. The statement

$$\textit{On } \Sigma, \ A \textit{ implies } B. \qquad (1)$$

means there is no element in Σ for which A is true and B is false, i.e., the cell marked by $*$ is empty. Similarly, "On Σ, B implies A." means there is no element in Σ for which B is true and A is false, or the cell marked by # is empty.

	B is true	B is false
A is true		$*$
A is false	#	

Table 2: Classification of Σ via truth of A (rows) and of B (columns).

If Σ is infinite, it is clearly not possible to classify each element in order to prove or disprove "A implies B.". Instead, some argument must be made for why the cell marked $*$ is empty, or an element must be shown to belong there, which is a counter-example. For instance, divisibility by 4 implies divisibility by 2 for all natural numbers, not just for S. This must be proved by showing that any multiple of 4 is also a multiple of 2. The argument is not difficult once you write down general expressions for multiples of 4 and of 2, and it is quite persuasive.

Table 1 shows that on S, divisibility by 2 does not imply divisibility by 4. Notice that none of the counter-examples $2, 6, 10, 14, 18, 22, 26, 30, 34, 38$ are perfect squares. Thus, on the subset of perfect squares in S, divisibility by 2 does imply divisibility by 4. In fact, Table 3 shows that both implications are true. We say divisibility by 2 and divisibility by 4 are **equivalent** on this set.

	Divisible by 4	Not divisible by 4
Divisible by 2	4, 16, 36	
Not divisible by 2		1, 9, 25

Table 3: Classification of perfect squares in S via divisibility by 2 (rows) and by 4 (columns).

Here is a generalisation for you to imagine a table and possibly construct a proof. Let p be a prime number. On the natural numbers which are perfect squares, divisibility by p implies divisibility by p^2.

In the case Σ consists of one element, there are only 4 possibilities for Table 2, of which exactly three correspond to (1) being true. For example, let

$$\Sigma = \{0\}, \quad A = \text{“... is greater than } -1.\text{”}, \quad B = \text{“... is less than 1.”}$$

Since both statements are true when applied to 0, on Σ, A implies B. But it is an odd statement, which is not helped by the paraphrase “If $0 > -1$, then $0 < 1$”. Even more bizarre is “If $0 > 1$, then $0 > 10$”. It appears that in order to make sense of (1), it is necessary that Σ is sufficiently large, so that A is true for some but not all elements, and likewise for B. The good news is that many important facts in mathematics are indeed so.

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