A Vacuous Theorem

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Given $n > 1$, let A and B be $n \times n$ matrices. Matrix multiplication is not commutative, i.e., generally $AB \neq BA$. However, there is a wonderful exception. If $AB = I_n$, the n-dimensional identity matrix, then it follows that $BA = I_n$. If this happens A and B are inverses of each other.

Let X be $n \times m$, where $n > m$. An $m \times n$ W is a left inverse of X if $WX = I_m$. An $m \times n$ Y is a right inverse of X if $XY = I_n$. Here is a theorem:

- Statement: Given $n > m$, let X be $n \times m$, and W, Y be its left and right inverses respectively. Then $W = Y$.

- Proof: $W = W(XY) = (WX)Y = Y$.

However, the theorem has no content: there are no matrices that satisfy the assumption. Specifically, it is impossible to find $n \times m$ X and $m \times n$ Y that satisfy $XY = I_n$.

This is an intriguing dilemma. On the one hand, we want to say a theorem is true. On the other hand, it seems odd, given that our theorem cannot be illustrated by any example. Perhaps there is a distinction between theorem and truth?