The Founding Orthogonal Projection

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For vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the dot product is $x \cdot y = \sum_{i=1}^n x_i y_i$, and the length of x is $|x| = \sqrt{x \cdot x}$. If $x \cdot y = 0$, x and y are orthogonal, written $x \perp y$. If there is a number c such that $y = cx$, x and y are collinear.

Let $\mathbf{1} = (1, \ldots, 1)$. Suppose $y \cdot \mathbf{1} \neq 0$, and y and $\mathbf{1}$ are not collinear. In other words, the entries of y do not sum to zero, and they are not all the same. Let their mean be $\overline{y}$.

Define $y_g = \kappa \mathbf{1}$ and $y_h = y - y_g$, where κ is a number such that $y_g \perp y_h$. y can be visualised as the hypotenuse of a right triangle, with y_g and y_h as the other sides. y_g is called the orthogonal projection of y on $\mathbf{1}$. Since $\kappa \neq 0$, the condition $(\kappa \mathbf{1}) \cdot (y - \kappa \mathbf{1}) = 0$ gives

$$\kappa = \frac{\mathbf{1} \cdot y}{\mathbf{1} \cdot \mathbf{1}} = \frac{\sum_{i=1}^n y_i}{n}$$

Thus, $y_g = (\overline{y}, \ldots, \overline{y})$, and $y_h = (y_1 - \overline{y}, \ldots, y_n - \overline{y})$ are the deviations of $y_1, \ldots, y_n$ from the mean. Moreover, we have the Pythagorean formula

$$|y|^2 = y_g \cdot y_g + 2 y_g \cdot y_h + y_h \cdot y_h = |y_g|^2 + |y_h|^2 \qquad \text{or} \qquad \sum_{i=1}^n y_i^2 = n\overline{y}^2 + \sum_{i=1}^n (y_i - \overline{y})^2$$

Dividing by n and rearranging yield the well-known fact that the variance of $y_1, \ldots, y_n$ is equal to the second moment minus the first moment squared. This can be derived by elementary algebra, but the orthogonal route seems more pleasant.

Happily, the orthogonal projection of y on any non-zero x which is neither orthogonal nor collinear to y is encapsulated in the special case above. Just replace $\mathbf{1}$ by x throughout to conclude the orthogonal projection of y on x is $\frac{x \cdot y}{x \cdot x} x$. This is a good foundation for generalising to orthogonal projection on a subspace of $\mathbb{R}^n$.

The elegant definition of orthogonality in $\mathbb{R}^n$ is motivated by the remarkable fact that perpendicular line segments in space correspond exactly to orthogonal vectors in $\mathbb{R}^3$. Similarly, the Pythagoreans would have agreed that the length of $x \in \mathbb{R}^3$ should be $|x|$. So the geometry of Euclid can be studied using coordinates, as discovered by Descartes and Fermat. Moreover, a triangle in any dimension can be visualised, and looks just like in $\mathbb{R}^3$. Indeed, $\mathbb{R}^n$ with the dot product is called the "n-dimensional Euclidean space".